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Multivariate Control Chart for Process Dispersion Based on Individual Observations

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Multivariate Control Chart for Process Dispersion Based on Individual Observations

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ABSTRACT

In this paper, we will discuss a simple way for monitoring shifts in the covariance matrix of a p-dimensional multivariate normal process distribution, $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. An exact method based on the chi-square distribution for constructing multivariate control limits will also be shown. We will illustrate the proposed procedure at work based on an example.

Key Words: Process dispersion; In-control; Out-of-control; Average run length (ARL).

INTRODUCTION

In most cases, the quality of an item from a production process is characterized by p correlated quality characteristics, $X_1, X_2, \ldots, X_p$. For example, the grade of lumber is often measured by its stiffness and bending strength (see Alt[1]). Approaches to multivariate quality control that exist today are based mainly on the Hotelling T^2 statistic that measures the significance of the shifted distance from the out-of-control mean vector, $\boldsymbol{\mu}_s$ to the nominal mean vector, $\boldsymbol{\mu}_0$. This approach is similar to the test of $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ vs. $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$, which specifies that the null hypothesis be rejected if

$$(\boldsymbol{X}_i - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}_0^{-1} (\boldsymbol{X}_i - \boldsymbol{\mu}_0) > \chi^2_{p,\alpha} \qquad (1)$$

where $\boldsymbol{\Sigma}_0$ is the nominal covariance matrix and $\boldsymbol{X}_i$, $i = 1, 2, \ldots,$ are independently and identically distributed (i.i.d.) observations from a multivariate normal, $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_0)$ process distribution (see Ref.[2]). In Eq. (1), it is assumed that the covariance matrix remains constant at $\boldsymbol{\Sigma}_0$ as we monitor for shifts in the mean vector, $\boldsymbol{\mu}$.

In practice, the assumption that the process dispersion remains constant at $\boldsymbol{\Sigma}_0$ must be validated. In the next section, a method for investigating this assumption is presented.

CONTROL CHART FOR PROCESS DISPERSION

The proposed multivariate control chart is based on the successive differences between multivariate observations.

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Its statistics are given in Eq. (2) below:

$$M_r = \frac{1}{2}(\boldsymbol{X}_r - \boldsymbol{X}_{r-1})'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{X}_r - \boldsymbol{X}_{r-1}),\quad r = 2, 3, \ldots \tag{2}$$

Here $\boldsymbol{X}_r$, $r = 2, 3, \ldots$ are i.i.d. random variables with a $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})$ process distribution. Both $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are the nominal mean vector and covariance matrix, respectively, where their values are either derived from a large amount of reliable past data or chosen by management to achieve certain objectives. Since M_r for $r = 2, 3, \ldots$ has a χ_p^2 distribution, which will be shown in the appendix, it follows that

$$\mathrm{P}(M_r > \chi^2_{p,\alpha}) = \alpha \tag{3}$$

and

$$\mathrm{P}(M_r < \chi^2_{p,1-\alpha}) = \alpha \tag{4}$$

where α is a specified Type-I error. This yields the upper control limit (UCL) for Eq. (3) as

$$\mathrm{UCL} = \chi^2_{p,\alpha} \tag{5}$$

and the lower control limit (LCL) for Eq. (4) as

$$\mathrm{LCL} = \chi^2_{p,1-\alpha} \tag{6}$$

where $\chi^2_{p,\alpha}$ is the 1-α percentile of the chi-square distribution with p degrees of freedom. The approach discussed here is similar to testing $H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ vs. $H_1 : \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$.

In general, Eqs. (3) and (5) are of more importance to quality-control practitioners since monitoring for an overall increase in the process dispersion is their major concern. For this case, the process is deemed to be out of control if the test statistic, M falls above $\mathrm{UCL} = \chi^2_{p,\alpha}$. However, Eq. (4) and (6), which are used to monitor for an overall decrease in the process dispersion, may provide information on process improvement, i.e., when the M statistic falls below $\mathrm{LCL} = \chi^2_{p,1-\alpha}$.

It should be noted that a two-sided control chart can also be constructed for detecting an overall shift in the process dispersion. For this case,

$$\mathrm{P}(\chi^2_{p,1-(\alpha/2)} < M_r < \chi^2_{p,\alpha/2}) = 1 - \alpha \tag{7}$$

The control limits are the upper control limit,

$$\mathrm{UCL} = \chi^2_{p,\alpha/2} \tag{8}$$

and the lower control limit,

$$\mathrm{LCL} = \chi^2_{p,1-(\alpha/2)} \tag{9}$$

If a value of the test statistic M_r, $r = 2, 3, \ldots$ falls above the UCL or below the LCL, then the process is deemed to be out of control.

SIMULATION STUDY AND RESULTS

In the simulation study, only the bivariate case (i.e., the number of quality characteristics, $p = 2$) is considered. However, the proposed procedure is also applicable to cases of $p > 2$. The used on-target mean vector is $\boldsymbol{\mu}_0 = (0, 0)'$ and the nominal covariance matrix is

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $\rho(0 \leq \rho \leq 1)$ is the correlation coefficient between the two quality characteristics. It will also be assumed that the process mean remains constant as the process dispersion shifts. To simulate shifts in the process dispersion, the following are considered:

(i) The standard deviations of both quality characteristics, X and Y, increase [$\delta > 1$] (or decrease [$\delta < 1$]) by the same magnitude, i.e., $\sigma_{0,X} \to \delta\sigma_{0,X}$ and $\sigma_{0,Y} \to \delta\sigma_{0,Y}$.

(ii) The standard deviation of one of the quality characteristics increases [$\delta > 1$] (or decreases [$\delta < 1$]), while the other one remains constant, i.e., $\sigma_{0,X} \to \delta\sigma_{0,X}$ and $\sigma_{0,Y}$ remains constant; or $\sigma_{0,X}$ remains constant and $\sigma_{0,Y} \to \delta\sigma_{0,Y}$.

The correlation coefficient (ρ) between both quality characteristics is assumed to be independent of the shifts in the standard deviations of at least one of the variables.

The performance of the proposed procedure is based on its average run length (ARL) performance. Computer programs for computing the ARL are written in the SAS programming language, version 6.12. Suppose that an in-control ARL of 200 is desired, then the UCL of Eq. (5) is $\chi^2_{2,0.005} = 10.595$ and the LCL of Eq. (6) is $\chi^2_{2,0.995} = 0.010025$.

The results for the case $\delta > 1$, based on the $\mathrm{UCL} = 10.595$ of Eq. (5) are given in Table 1 and for the case $\delta < 1$ based on the $\mathrm{LCL} = 0.010025$ of Eq. (6), are given in Table 2. The in-control ARLs in both Tables 1 and 2 are slightly above the expected value 200. A possible reason is the test statistic M of Eq. (2) can only

Table 1. Simulation results for a proposed multivariate control chart for process dispersion when $\delta > 1$.

δ	$\sigma_{0,X} \to \delta\sigma_{0,X}$ $\sigma_{0,Y} \to \delta\sigma_{0,Y}$	$\sigma_{0,X} \to \delta\sigma_{0,X}$ $\sigma_{0,Y} \to \sigma_{0,Y}$	$\sigma_{0,X} \to \sigma_{0,X}$ $\sigma_{0,Y} \to \delta\sigma_{0,Y}$
1.0	208.4	208.4	208.4
1.1	85.2	123.3	124.7
1.2	45.0	76.5	77.4
1.3	26.7	50.9	51.4
1.5	13.1	26.7	27.0
2.0	5.4	10.3	10.2
2.5	3.6	6.3	6.2
3.0	2.9	4.8	4.7
4.0	2.4	3.5	3.5
6.0	2.2	2.7	2.7
10.0	2.1	2.3	2.3

Table 2. Simulation results for a proposed multivariate control chart for process dispersion when $\delta < 1$.

δ	$\sigma_{0,X} \to \delta\sigma_{0,X}$ $\sigma_{0,Y} \to \delta\sigma_{0,Y}$	$\sigma_{0,X} \to \delta\sigma_{0,X}$ $\sigma_{0,Y} \to \sigma_{0,Y}$	$\sigma_{0,X} \to \sigma_{0,X}$ $\sigma_{0,Y} \to \delta\sigma_{0,Y}$
1.0	202.0	202.0	202.0
0.9	163.3	182.0	181.1
0.8	128.4	161.8	160.8
0.7	98.0	140.7	141.3
0.5	50.3	100.0	99.8
0.3	19.5	61.5	59.5
0.1	3.7	23.4	23.6
0.05	2.2	15.9	16.0
0.025	2.0	14.2	14.1

be computed from the second observation onward and not the first. From Table 1, it is shown that as the increase in standard deviation for one or both quality characteristics becomes larger, the sensitivity of the chart based on Eqs. (2), (3), and (5) increases. Similarly, from Table 2, the sensitivity of the proposed control chart based on Eqs. (2), (4), and (6) increases as the decreases in standard deviation for one or both quality characteristics become larger.

EXAMPLE

In this section, an example will be given on how we can put the proposed control chart to work. This numerical example is based on data generated from SAS version 6.12. We have generated 16 observations from a bivariate normal distribution, $N_2(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, followed by observations 17 to 22 from a $N_2(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_s)$ distribution where

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 300 \\ 400 \end{pmatrix}, \boldsymbol{\Sigma}_0 = \begin{pmatrix} 100 & 72 \\ 72 & 144 \end{pmatrix},$$

$$\boldsymbol{\Sigma}_s = \begin{pmatrix} 256 & 172.8 \\ 172.8 & 324 \end{pmatrix}$$

and $\rho = 0.6$. Here, we assume that the value of ρ is stable with respect to shifts in the process dispersion. Suppose that the standard deviation of the first variable (x_1) increases by 60% and the second (x_2) by 50%, then the shifted covariance matrix, $\boldsymbol{\Sigma}_s$ is obtained as follows:

$$\boldsymbol{\Sigma}_s = \begin{pmatrix} (1.6)^2 \times 100 & 0.6 \times \sqrt{(1.6)^2 \times 100} \times \sqrt{(1.5)^2 \times 144} \\ 0.6 \times \sqrt{(1.6)^2 \times 100} \times \sqrt{(1.5)^2 \times 144} & (1.5)^2 \times 144 \end{pmatrix}$$

$$= \begin{pmatrix} 256 & 172.8 \\ 172.8 & 324 \end{pmatrix}$$

The first 16 observations represent the stable (in-control) process distribution, whereas the last 6 observations represent the out-of-control process distribution. The 22 observations generated are substituted in Eq. (2) to compute the corresponding M statistics. Table 3 shows the values of variables x_1, x_2 and the corresponding M statistics.

Suppose that in this example we wish to monitor for an increase in the process dispersion, then, from Eqs. (3) and (5), the process is deemed out of control if the test statistic, M falls above $\text{UCL} = \chi^2_{p,\alpha}$. If the Type-I error ($\alpha$) is set equal to 0.005, then UCL = 10.60. From Table 3, the M statistics for observations 18, 21, and 22 are all greater than 10.60. Note also that the chart signals an out-of-control at observation 18.

CONCLUSION

The proposed procedure can be a useful tool in process control situations involving simultaneous

Table 3. *M* Statistics for a proposed multivariate control chart for process dispersion.

Observation number	x_1	x_2	M_r
1	294.910	396.523	—
2	288.005	379.919	0.973
3	296.515	414.298	4.692
4	289.119	399.655	0.745
5	297.852	403.488	0.414
6	297.768	390.059	0.970
7	288.279	395.104	1.216
8	312.510	414.455	2.955
9	300.118	423.022	2.427
10	315.033	401.014	6.930
11	304.072	400.938	0.932
12	299.442	392.974	0.223
13	295.712	394.946	0.187
14	310.241	403.880	1.068
15	294.882	385.110	1.502
16	321.499	411.282	3.809
17	336.544	428.451	1.350
18	283.331	384.590	14.325*
19	277.561	382.888	0.199
20	296.515	397.914	1.807
21	347.612	424.519	13.617*
22	281.199	370.972	22.232*

monitoring of several quality characteristics. Eqs. (2) to (9) of the proposed procedure can be used to study the state of variability of a production process involving more than one quality characteristic as is illustrated in the above example. Since the exact distribution of the test statistic, *M*, for the stable process is known, management can set the control limits based on the desired in-control ARL, hence, providing ease of use.

APPENDIX

Let $X_1, X_2, \ldots, X_r, \ldots$ be a sequence of i.i.d. random variables with a $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ process distribution, where both $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are the nominal mean vector and covariance matrix, respectively.

Then

$$\boldsymbol{X}_r - \boldsymbol{X}_{r-1} \sim N_p(0, 2\boldsymbol{\Sigma}_0), \quad r = 2, 3, \ldots$$

and

$$\frac{1}{\sqrt{2}}(\boldsymbol{X}_r - \boldsymbol{X}_{r-1}) \sim N_p(0, \boldsymbol{\Sigma}_0), \ r = 2, 3, \ldots$$

Let $CC' = \boldsymbol{\Sigma}_0$, where C is a nonsingular $p \times p$ matrix. If one defines the statistic

$$\boldsymbol{U} = \frac{1}{\sqrt{2}} C^{-1}(\boldsymbol{X}_r - \boldsymbol{X}_{r-1})$$

then

$$\boldsymbol{U}'\boldsymbol{U} = \frac{1}{2}(\boldsymbol{X}_r - \boldsymbol{X}_{r-1})'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{X}_r - \boldsymbol{X}_{r-1})$$

is a sum of squares of p independent $N(0,1)$ variables; hence, $\boldsymbol{U}'\boldsymbol{U} = \boldsymbol{M}_r \sim \chi_p^2$.

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