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## **Asymmetric Multivariate Stochastic Volatility**

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## ASYMMETRIC MULTIVARIATE STOCHASTIC VOLATILITY

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□ *This paper proposes and analyses two types of asymmetric multivariate stochastic volatility (SV) models, namely, (i) the SV with leverage (SV-L) model, which is based on the negative correlation between the innovations in the returns and volatility, and (ii) the SV with leverage and size effect (SV-LSE) model, which is based on the signs and magnitude of the returns. The paper derives the state space form for the logarithm of the squared returns, which follow the multivariate SV-L model, and develops estimation methods for the multivariate SV-L and SV-LSE models based on the Monte Carlo likelihood (MCL) approach. The empirical results show that the multivariate SV-LSE model fits the bivariate and trivariate returns of the S&P 500, the Nikkei 225, and the Hang Seng indexes with respect to AIC and BIC more accurately than does the multivariate SV-L model. Moreover, the empirical results suggest that the univariate models should be rejected in favor of their bivariate and trivariate counterparts.*

**Keywords** Asymmetric leverage; Bayesian Markov chain Monte Carlo; Dynamic leverage; Importance sampling; Multivariate stochastic volatility; Numerical likelihood; Size effect.

**JEL Classification** C15; C32.

### 1. INTRODUCTION

In both the conditional and stochastic volatility (SV) literature, there has been some confusion regarding the definitions of asymmetry and leverage. Originally, Christie (1982) investigated the negative relation between the ex-post volatility in the rate of returns on equity and the current value of the equity. We will refer to this phenomenon as the “leverage” effect. On the other hand, the “asymmetric” effect in volatility means that the effects of positive returns on volatility are

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different from those of negative returns of a similar magnitude. Therefore leverage denotes asymmetry, but not all asymmetric effects display leverage. In the class of ARCH specifications that have been developed to capture asymmetric effects, the Exponential GARCH (EGARCH) model of Nelson (1991) and the GJR model of Glosten et al. (1993) are widely used. Using the terminology given above, the EGARCH model can describe leverage, whereas the GJR model can capture asymmetric effects but not leverage (for further details, see Asai and McAleer, 2005, to appear).

The asymmetric property of the SV model is based on the direct correlation between the innovations in both returns and volatility. For a theoretical development in the continuous time framework, Hull and White (1987) generalized the Black–Scholes option pricing formula to analyze SV and the negative correlation between the innovation terms. In empirical research, extensions of a simple discrete time model due to Taylor (1986) have been analyzed by Wiggins (1987), Chesney and Scott (1989), and Harvey and Shephard (1996) in order to accommodate the direct correlation. Although this extension has been called the asymmetric SV model, we will refer to the asymmetric behavior based on the direct correlation between the innovations as the “SV with leverage” (SV-L) model to distinguish it from an alternative model of asymmetry to be discussed below.

Danielsson (1994) suggested an alternative type of asymmetric SV model that is similar in spirit to that of the EGARCH model. Nelson (1991) used the absolute value function to capture the sign and magnitude of the previous value of normalized returns in accommodating asymmetric behavior into an ARCH-type model. Danielsson (1994) used the absolute value function as in Nelson (1991) but incorporated the observed return into the SV specification, as it is not computationally straightforward in the SV framework to incorporate the normalized disturbances. For this reason, we will refer to this type of specification as the “SV with leverage and size effect” (SV-LSE) model.

Recently, So et al. (2002) considered a different type of threshold effects model in which the breaks in the constant and autoregressive parameter in the SV equation depend on the signs of the previous returns. An alternative form of asymmetry can be based on threshold effects, as proposed in Glosten et al. (1993) in the context of conditional volatility models. These models will not be discussed in detail here as the empirical results in Asai and McAleer (2005) show that their model is generally inferior to the SV-L model in terms of AIC and BIC. A variety of symmetric and asymmetric, univariate and multivariate, conditional and stochastic volatility models is analyzed in McAleer (2005).

The first multivariate SV model was proposed by Harvey et al. (1994), who specified the model in terms of instantaneous correlations in the mean and volatility equations. However, their estimation technique was

based on the inefficient quasi-maximum likelihood (QML) procedure. Danielsson (1998) suggested a multivariate SV-L model based on the specification considered by Harvey et al. (1994) but only estimated a symmetric version of the model. Shephard (1996) proposed a one-factor multivariate SV model, while Liesenfeld and Richard (2003) proposed an efficient importance sampling method and estimated the one-factor model.

This paper considers multivariate extensions of the SV-L and SV-LSE models. The SV-L model, which is considered in Danielsson (1998), assumes a negative correlation between returns and volatility innovations. As an alternative, the SV-LSE model accommodates the effect of the sign and magnitude of the previous return in the volatility equation by using the absolute value function. Both models assume instantaneous correlations in the mean and volatility equations.

In order to estimate these multivariate models, this paper employs the numerical or Monte Carlo likelihood (MCL) method proposed by Durbin and Koopman (1997). Sandmann and Koopman (1998) and Koopman and Uspensky (2002) used the MCL method to estimate the SV-L model and the SV in mean model (without asymmetry), respectively. The efficiency of the MCL method is similar to the Bayesian Markov chain Monte Carlo (MCMC) method proposed by Jacquier et al. (1994). However, for the problem considered in this paper, the computational burden of the MCL method is about one-fifth of the MCMC method, and one-eighth of Danielsson's (1994) accelerated Gaussian importance sampling (AGIS) approach.

The paper is organized as follows. Section 2 discusses univariate and multivariate asymmetric SV models and presents a state space form for the logarithm of the squared returns that follow the multivariate SV-L model. Section 3 discusses the MCL method and develops an extension for estimating multivariate SV models. Section 4 presents some empirical results by using trivariate data of Standard and Poor's 500 Composite Index, the Nikkei 225 Index, and the Hang Seng Index. Section 5 presents some concluding remarks.

In the following section,  $\exp(\cdot)$  and  $\ln(\cdot)$  denote the element-by-element exponential and logarithmic operators, respectively, and  $\text{diag}\{x\} = \text{diag}\{x_1, \dots, x_m\}$  denotes the  $M$ -dimensional diagonal matrix, with diagonal elements given by  $x = (x_1, \dots, x_m)'$ .

## 2. ASYMMETRIC STOCHASTIC VOLATILITY MODELS

Alternative univariate and multivariate asymmetric SV-L and SV-LSE models will be considered in this section.

## 2.1. Univariate Models

The SV-L model captures asymmetry through the negative correlation between the returns and volatility innovations, as

$$y_t = \sigma \varepsilon_t \exp\left(\frac{h_t}{2}\right), \quad (1)$$

$$\varepsilon_t \sim N(0, 1), \quad t = 1, \dots, T,$$

$$h_{t+1} = \mu + \phi h_t + \eta_t, \quad (2)$$

$$\eta_t \sim N(0, \sigma_\eta^2),$$

$$E(\varepsilon_t \eta_t) = \lambda \sigma_\eta,$$

where  $y_t = R_t - E(R_t | \mathfrak{F}_{t-1})$  and  $R_t$  is the return on a financial asset. For purposes of identification, it is necessary to set either  $\sigma = 1$  or  $\mu = 0$ . This paper uses the restriction  $\sigma = 1$ , as it is preferable for comparing the SV-LSE model with the model of Danielsson (1994) shown below. As discussed in Chesney and Scott (1989), this model has an interpretation of some continuous time models.

Conditionally on the signs of  $y_t$ , Harvey and Shephard (1996) showed that the statespace form for the logarithm of squared returns,  $x_t = \ln y_t^2$ , was given as

$$x_t = h_t + \zeta_t, \quad (3)$$

$$\zeta_t = \ln \varepsilon_t^2 \sim \ln \chi^2(1),$$

$$h_{t+1} = \mu + A s_t + \phi h_t + \eta_t^*, \quad (4)$$

$$\eta_t^* \sim N(0, \sigma_\eta^2 - A^2),$$

$$\text{Cov}(\zeta_t, \eta_t^*) = B s_t,$$

where  $s_t$  takes the value one (minus one) if  $y_t$  is positive (otherwise),  $A = 0.7979\lambda\sigma_\eta$ , and  $B = 1.1061\lambda\sigma_\eta$ . The mean and variance of  $\zeta_t$  are  $-1.2703$  and  $\pi^2/2$ , respectively. Harvey and Shephard (1996) proposed the quasi-maximum likelihood (QML) method of estimating the model based on the Kalman filter. The normal approximation of  $\ln \chi^2(1)$ , which is far from being normal, implies that the QML estimator is likely to have poor small sample properties, even though it is consistent.

In order to cope with this problem, Sandmann and Koopman (1998) suggested the Monte Carlo likelihood (MCL) method, which will be explained in Section 3. Asai and McAleer (2005) conducted Monte Carlo experiments to investigate the finite sample properties of the MCL

estimator. It was shown that the bias in the MCL estimator was generally very small, and that the coverage probability (or the fraction of times that the true parameter value falls within the confidence interval) was close to the true value.

It may be useful to note two other estimation methods, namely the efficient method of moments (EMM) method proposed by Gallant and Tauchen (1996), and the Bayesian MCMC method of Jacquier et al. (2004) and Yu (2005). The EMM matches the scores of an auxiliary model via simulation. Gallant and Tauchen (1996) stated that, if the auxiliary model is a good approximation to the distribution of the data, the EMM estimator is as efficient as maximum likelihood. Chernov et al. (2003) used the EMM approach to estimate asymmetric SV models in the more general framework. However, the EMM provides no estimate of the instantaneous volatility, so that an additional form of estimation is required. Second, compared with the MCL method of Sandmann and Koopman (1998), the Bayesian MCMC method is computationally demanding. The Monte Carlo results of Sandmann and Koopman (1998), which compare the MCL method with the MCMC method of Jacquier et al. (1994), show that MCL yields a larger bias than MCMC when the unconditional variance of the time-varying log-volatility is relatively small. This outcome is not particularly relevant for the data set used in this paper, as such a result suggests that the volatility is not particularly significant.

This paper focuses on another asymmetric type of SV model. Danielsson (1994) considered the model

$$y_t = \varepsilon_t \exp(h_t/2), \quad \varepsilon_t \sim N(0, 1), \quad t = 1, \dots, T, \quad (5)$$

$$h_{t+1} = \mu + \gamma_1 y_t + \gamma_2 |y_t| + \phi h_t + \eta_t, \quad \eta_t \sim N(0, \sigma_\eta^2), \quad (6)$$

where  $E(\varepsilon_s \eta_t) = 0$  for any  $s$  and  $t$ . This model incorporates the effect of the sign and magnitude of the previous return into the volatility equation by using the absolute value function. We refer to this type of asymmetry as an “SV with leverage and size effect” (SV-LSE) model. In the original work of Danielsson (1994), the weak serial correlation of stock returns was also considered in the SV model. In the current paper, we will treat such effects by using the definition  $y_t = R_t - E(R_t | \mathfrak{F}_{t-1})$ . It is possible to estimate the model using the accelerated Gaussian importance sampler (AGIS) algorithm, which is a simulation-based technique with time requirements and precision that are similar to those of the MCMC method. However, the AGIS method is difficult to generalize to multivariate SV-L models and remains computationally demanding relative to the MCL method.

Asai and McAleer (2005) estimated the SV-L and SV-LSE models by using returns of S&P 500 and TOPIX stock index returns, and the AUD/USD and Japanese Yen/USD exchange rates. The empirical results

showed that all four data sets always preferred the SV-LSE model to the SV-L model on the basis of both AIC and BIC.

## 2.2. Multivariate Models

Danielsson (1998) considered a multivariate extension of the SV-L model. The matrix representation of the model is given by

$$y_t = D_t \varepsilon_t, \quad (7)$$

$$D_t = \text{diag}\{e^{h_{1t}/2}, \dots, e^{h_{mt}/2}\} = \text{diag}\{\exp(0.5h_t)\},$$

$$h_{t+1} = \mu + \phi \circ h_t + \eta_t, \quad (8)$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \sim N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} P_\varepsilon & L \\ L & \Sigma_\eta \end{pmatrix} \right], \quad (9)$$

$$L = \text{diag}\{\lambda_1 \sigma_{11}^{1/2}, \dots, \lambda_m \sigma_{mm}^{1/2}\},$$

where  $h_t = (h_{1t}, \dots, h_{mt})'$  is the vector of unobserved log-volatility,  $\mu$  and  $\phi$  are  $m \times 1$  parameter vectors, the operator  $\circ$  denotes the Hadamard (or element-by-element) product,  $\Sigma_\eta = \{\sigma_{\eta,ij}\}$  is the positive-definite covariance matrix, and  $P_\varepsilon = \{\rho_{ij}\}$  is the correlation matrix, such that  $P_\varepsilon$  is a positive definite matrix with  $\rho_{ii} = 1$  and  $|\rho_{ij}| < 1$  for any  $i \neq j$ . When  $\lambda_1 = \dots = \lambda_m = 0$ , the multivariate SV-L model reduces to the model of Harvey et al. (1994). If the off-diagonal elements of  $P_\varepsilon$  and  $\Sigma_\eta$  are all equal to zero, each element of  $y_t$  follows the univariate SV-L model. In other words, all assets affect each other through the correlation matrix of the conditional distribution and/or the covariance matrix of the log-volatility process.

Assuming that the off-diagonal elements of  $\Sigma_\eta$  are all equal to zero, the model corresponds to the constant conditional correlation (CCC) model proposed by Bollerslev et al. (1988) in the framework of multivariate GARCH models. In the CCC model, each conditional variance is specified as a univariate GARCH model (that is, with no spillovers from any other asset), while each conditional covariance assumes a constant conditional correlation times the corresponding conditional standard deviations. Thus if the off-diagonal elements of  $\Sigma_\eta$  are not all equal to zero, then the elements of  $h_t$  have spillover effects across assets. It should be noted that Danielsson (1998) only suggested the multivariate SV-L model given in Equations (7)–(9) but did not estimate the model.

Conditionally on the signs of each element of  $y_t$ , the logarithmic transformation of each element of  $y_t^2$  gives

$$x_t = \ln y_t^2 = h_t + \xi_t, \quad (10)$$

$$\xi_t = \ln \varepsilon_t^2,$$

$$\begin{aligned}
 h_{t+1} &= \mu + \mu_t^* + \phi \circ h_t + \eta_t^*, \\
 \eta_t^* &\sim N(0, \Sigma_{\eta,t}^*), \\
 E(\eta_t^* \zeta_t') &= L_t^*, \\
 \Sigma_{\eta,t}^* &= \Sigma_{\eta} - LP_{\varepsilon}^{-1}L + LP_{\varepsilon}^{-1} \left[ \left\{ P_{|\varepsilon|} - \frac{2}{\pi} tt' \right\} \circ (s_t s_t') \right] P_{\varepsilon}^{-1}L, \\
 \mu_t^* &= \sqrt{\frac{2}{\pi}} LP_{\varepsilon}^{-1} s_t, \quad L_t^* = LP_{\varepsilon}^{-1} \left[ \left\{ R_{|\varepsilon|} - c \sqrt{\frac{2}{\pi}} \right\} \circ (s_t t') \right],
 \end{aligned}
 \tag{11}$$

where  $s_t = (s_{1t}, \dots, s_{mt})'$  and  $s_{it}$  takes one (minus one) if  $y_{it}$  is positive (otherwise),  $L$  and  $P_{\varepsilon}$  are defined by (9), and  $P_{|\varepsilon|}$  and  $R_{|\varepsilon|}$  are given in the appendix (where Equations (10) and (11) are also derived). When  $\lambda_1 = \dots = \lambda_m = 0$ , that is,  $L = 0$ , this state space form reduces to the model of Harvey et al. (1994). If there is no correlation across the variables, that is,  $P_{\varepsilon} = I_m$  and  $\Sigma_{\eta} = \text{diag}\{\sigma_{\eta,11}, \dots, \sigma_{\eta,mm}\}$ , each  $y_{it}$  has the state space form derived in Harvey et al. (1994).

The mean of each element of  $\zeta_t$  is  $-1.2703$ . Harvey et al. (1994) showed that the covariance matrix of  $\zeta_t$ , denoted  $\Sigma_{\zeta}$ , is given by  $\Sigma_{\zeta} = (\pi^2/2)\{\rho_{ij}^*\}$ , where  $\rho_{ii}^* = 1$ ,

$$\rho_{ij}^* = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(n-1)!}{(1/2)_n n} \rho_{ij}^{2n}, \tag{12}$$

and  $(x)_n = x(x+1)\dots(x+n-1)$ . If  $\rho_{ij}^*$  can be estimated, then it is also possible to estimate the absolute value of  $\rho_{ij}$ , and the cross correlation between different values of  $\varepsilon_{it}$ . Estimation of the signs of  $\rho_{ij}$  may be obtained by returning to the untransformed observations, and noting that the sign of each of the pairs  $\varepsilon_{it}\varepsilon_{jt}$  ( $i, j = 1, \dots, m$ ) will be the same as for the corresponding pairs of observed values,  $y_{it}y_{jt}$ . Thus the sign of  $\rho_{ij}$  is estimated as positive if more than one-half of the pairs  $y_{it}y_{jt}$  is positive.

By using the density function of  $\ln \chi^2(1)$  given in Sandmann and Koopman (1998) and the linear transformation, we have the density function of  $\zeta_t$  as

$$f(x) = (2\pi)^{-m/2} |P_{\zeta}|^{-1/2} \exp \left[ \frac{1}{2} \left\{ c(m - t'P_{\zeta}^{-1/2}t) + t'P_{\zeta}^{-1/2}x - k(x) \right\} \right],$$

where  $P_{\zeta} = \{\rho_{ij}^*\}$ ,  $c = -1.2703$ ,  $t$  is an  $m \times 1$  vector of ones,

$$k(x) = \sum_{i=1}^m \exp(c + q_i(x - ct)), \tag{13}$$

and  $q_i$  is the  $i$ th row of  $P_\xi^{-1/2}$ . If there is no correlation among the variables, that is,  $P_\xi = I_m$ , then the density function reduces to a multiple of the density function of  $\ln \chi^2(1)$ . In Section 3, this paper develops an MCL method to estimate the multivariate SV-L model based on the true density function and the state space form given in Equations (10) and (11).

The SV-LSE model can be extended to the multivariate case in a similar way, as

$$y_t = D_t \varepsilon_t, \quad (14)$$

$$D_t = \text{diag}\{e^{h_{1t}/2}, \dots, e^{h_{mt}/2}\} = \text{diag}\{\exp(0.5h_t)\}, \quad (15)$$

$$h_{t+1} = \mu + \gamma_1 \circ \gamma_1 + \gamma_2 \circ |\gamma_1| + \phi \circ h_t + \eta_t, \quad (16)$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \sim N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} P_\varepsilon & 0 \\ 0 & \Sigma_\eta \end{pmatrix} \right], \quad (17)$$

where  $\gamma_1$  and  $\gamma_2$  are  $m \times 1$  parameter vectors. When the off-diagonal elements of  $P_\varepsilon$  and  $\Sigma_\eta$  are all equal to zero, each element of  $y_t$  follows an independent univariate SV-LSE model. If the off-diagonal elements of  $\Sigma_\eta$  are all equal to zero, then the model corresponds to the SV version of the CCC model of Bollerslev et al. (1988). Section 3 will explain the MCL method to estimate the multivariate SV-LSE model.

### 3. MONTE CARLO MAXIMUM LIKELIHOOD ESTIMATION

This section develops the MCL method to estimate two types of multivariate asymmetric SV models, namely the multivariate SV-L and SV-LSE models. The first part of this section briefly explains the general framework of the MCL approach proposed by Durbin and Koopman (1997). The remainder of this section constructs the approximating densities, as required by the MCL approach, for each of the multivariate SV-LSE and SV-L models. The MCL is based on the density of  $x_t = \ln y_t^2$ .

#### 3.1. Likelihood Evaluation Via Importance Sampling

For the MCL method, the likelihood function can be approximated arbitrarily by decomposing it into a Gaussian part, which is constructed by the Kalman filter, and a remainder function, for which the expectation is evaluated through simulation.

Let  $x = (x_1, \dots, x_T)'$  and  $h = (h_1, \dots, h_T)'$ , and denote the marginal densities of  $x$  and  $h$ , their joint density, and the conditional density of  $x$  given  $h$  for a given unknown parameter vector  $\psi$ , by  $p(x|\psi)$ ,

$p(h|\psi)$ ,  $p(x, h|\psi)$ , and  $p(x|h, \psi)$ , respectively. The likelihood function is defined by

$$L(\psi) = p(x|\psi) = \int p(x, h|\psi) dh = \int p(x|h, \psi)p(h|\psi) dh. \quad (18)$$

Durbin and Koopman (1997) considered the likelihood of the approximating the Gaussian model as

$$L_g(\psi) = g(x|\psi) = \frac{g(x|h, \psi)p(h|\psi)}{g(h|x, \psi)}.$$

Note that the MCL method uses the same density of  $h|\psi$  as the true model to construct the approximating Gaussian model. Substituting  $p(h|\psi)$  from the above equation into (18) gives

$$L(\psi) = \int L_g(\psi) \frac{p(x|h, \psi)}{g(x|h, \psi)} g(h|x, \psi) dh = L_g(\psi) E_g \left[ \frac{p(x|h, \psi)}{g(x|h, \psi)} \right],$$

where  $E_g$  denotes the expectation with respect to  $g(h|x, \psi)$ . The advantage of the approach of Durbin and Koopman (1997) is that it requires simulation only to estimate departures of the likelihood from the Gaussian likelihood, rather than the likelihood itself. Durbin and Koopman (1997) suggested that  $g(h|x, \psi)$  be employed as the importance density for the simulations.

The log-likelihood function is given by

$$\ln L(\psi) = \ln L_g(\psi) + \ln E_g \left[ \frac{p(x|h, \psi)}{g(x|h, \psi)} \right], \quad (19)$$

and its consistent estimator is

$$\ln \widehat{L}(\psi) = \ln L_g(\psi) + \ln \bar{w} + \frac{\bar{w}^2}{2N s_w^2}, \quad (20)$$

where  $N$  is the number of simulations,

$$\bar{w} = \frac{1}{N} \sum_{i=1}^N w_i, \quad s_w^2 = \frac{1}{N-1} \sum_{i=1}^N (w_i - \bar{w})^2, \quad w_i = \frac{p(x|h^{(i)}, \psi)}{g(x|h^{(i)}, \psi)},$$

and  $h^{(i)}$  denotes a draw from the importance density  $g(h|y, \psi)$  (see Durbin and Koopman, 1997; Koopman and Uspensky, 2002; Sandmann and Koopman, 1998, for detailed discussions of the method).

The MCL method obtains estimates of the parameters,  $\psi$ , through numerical optimization of Equation (20). The log-likelihood function of

the approximating model,  $\ln L_g(\psi)$ , can be used to obtain the starting values. The choice of  $N$  governs the accuracy of the approximation to the likelihood function such that, as  $N$  increases, the approximation becomes more accurate. All the calculations in this paper are based on  $N = 200$ .

Sandmann and Koopman (1998), Koopman and Uspensky (2002), and Asai and McAleer (2005) investigated finite sample properties of an MCL estimator for univariate SV models. As stated in the previous section, these authors showed that the MCL method is useful practically for estimating various kinds of SV models.

### 3.2. Approximating Gaussian Density for Asymmetric Multivariate SV Models

For convenience, this paper first constructs the approximating Gaussian density for the multivariate SV-L model. As the transformed multivariate SV-L model has the state space form given by (10) and (11), the approximating Gaussian density is based on the linear Gaussian model given by

$$\begin{aligned} x_t &= \ln y_t^2 = h_t + u_t, \\ u_t &\sim N(\mu_t, H_t), \quad t = 1, \dots, T, \end{aligned} \quad (21)$$

where  $\mu_t$  and  $H_t$  are selected in such a way that the time-varying mean and variance of  $x_t$  implied by the approximating model (21) are as close as possible to the true model given by (10) and (11).

As for the non-Gaussian true density given by  $p(\xi_t | \psi)$ , we have

$$p(x_t | h_t, s_t, \psi) = p(x_t - h_t | s_t, \psi) = p(\xi_t | s_t, \psi).$$

Based on this fact, it is possible to obtain  $\mu_t$  and  $H_t$  by equalizing the first and second derivatives of  $p(x_t | h_t, s_t, \psi)$  and the approximating density with respect to  $\xi_t$ , as

$$\begin{aligned} -\frac{\partial \ln p(\xi_t | s_t, \psi)}{\partial \xi_t} \Big|_{\xi_t = \hat{\xi}_t} - H_t^{-1}(\xi_t - \mu_t) &= 0, \\ -\frac{\partial^2 \ln p(\xi_t | s_t, \psi)}{\partial \xi_t \partial \xi_t'} \Big|_{\xi_t = \hat{\xi}_t} + H_t^{-1} &= 0, \end{aligned}$$

where  $\hat{\xi}_t = E_g(\xi_t | x, s, \psi)$  is obtained from the Kalman filter and smoother. Although this approach yields a positive definite matrix  $H_t$ , as

$$\left\{ \frac{1}{2} \frac{\partial^2 k(\xi)}{\partial \xi \partial \xi'} \right\}^{-1} \Big|_{\xi_t = \hat{\xi}_t} = \left\{ \sum_{i=1}^m \exp(q_i \xi) q_i' q_i \right\}^{-1} \Big|_{\xi = \hat{\xi}_t},$$

where  $k(\cdot)$  and  $q_i$  are defined by Equation (13), this matrix is not stable. Instead of this covariance matrix, it is suggested in this paper that we use

$$H_t = Q_t^{1/2} P_\xi Q_t^{1/2}, \quad (22)$$

where

$$Q_t = \text{diag} \left\{ \frac{2z_{1t}}{e^{z_{1t}} - 1}, \dots, \frac{2z_{mt}}{e^{z_{mt}} - 1} \right\},$$

$$z_{it} = c + q_i (\hat{\xi}_{it} - c),$$

in order to yield

$$\mu_t = \hat{\xi}_t + 0.5 H_t \left[ \{P_\xi^{-1/2}\}'_t - \sum_{i=1}^m \exp(c + q_i (\hat{\xi}_t - c)) q_i' \right].$$

These  $\mu_t$  and  $H_t$  can be interpreted as natural extensions of the approach of Sandmann and Koopman (1998) in that they reduce to the original values proposed in the MCL method when  $m = 1$ , namely 0 and  $2\hat{\xi}_t / (e^{\hat{\xi}_t} - 1)$  for  $\mu_t$  and  $H_t$ , respectively. It should be noted that  $\hat{\xi}_t = E_g(\xi_t | x, s, \psi)$  is obtained from the Kalman filter and smoother. This procedure usually requires 7–9 iterations before convergence at each step of the optimization.

In addition to  $H_t$ , it is necessary to guarantee the positive definiteness of

$$\Omega_t = \begin{pmatrix} H_t & (L_t^*)' \\ L_t^* & \Sigma_{\eta t}^* \end{pmatrix},$$

where  $L_t^*$  and  $\Sigma_{\eta t}^*$  are defined by Equation (11), in order to perform the Kalman filter and smoother. Asai (2005) suggested using the nearest covariance matrix proposed by Higham (1988), who proved that the nearest positive symmetric matrix in the Frobenius norm to any real symmetric matrix  $C$  is  $(C + P)/2$ , where  $P$  is the symmetric polar factor of  $C$ . When  $\Omega_t$  is nonpositive semidefinite in the approximating density, this approach replaces  $\Omega_t$  by its nearest covariance matrix.

Now consider the approximating Gaussian density for the multivariate SV-LSE models, for which the approximating model is given by Equations (16) and (21). Thus we can apply the same approach stated above except for the nearest covariance matrix. As  $L = 0$  in this case,  $\Omega_t$  is always positive definite.

#### 4. EMPIRICAL RESULTS

This section examines the MCL estimates of the univariate and multivariate SV-L and SV-LSE SV models for three sets of empirical data,

namely Standard and Poor’s 500 Composite Index (S&P), the Nikkei 225 Index (Nikkei), and the Hang Seng Index (Hang Seng). The sample period for all three series is 1/2/1986 to 10/4/2000, giving  $T = 3605$  observations. Returns  $R_{it}$  are defined as  $100 \times \{\log P_{it} - \log P_{i,t-1}\}$ , minus the sample mean, where  $P_{it}$  is the closing price on day  $t$  for stock  $i$ . The autocorrelation structure in the stock returns was removed by using the threshold AR(1) model

$$E(R_{it} | \mathfrak{F}_{t-1}) = c_{i,d_{t-1}} + \theta_{i,d_{t-1}} R_{i,t-1},$$

where  $d_{it}$  is 0 if  $R_{it} > 0$  and 1 otherwise. Hereafter, for convenience, we will refer to the stock returns as  $\hat{y}_{it} = R_{it} - \widehat{E}(R_{it} | \mathfrak{F}_{t-1})$ .

Table 1 shows the MCL estimates of the univariate SV-L and SV-LSE models. All the estimated parameters are significant at the five percent level, except for  $\mu$  in the SV-L model for the Nikkei. As Table 1 also shows that  $\rho < 0$  in the SV-L model and  $\gamma_1 < \gamma_2 < 0$  in the SV-LSE model, there exist clear leverage effects in all three data sets. The SV-L and SV-LSE models are estimated by using the MCL method based on the distribution of  $\ln y_t^2$ . For all data sets, the SV-LSE model is preferred to the SV-L model on the basis of AIC and BIC.

Table 2 presents the estimates of the bivariate SV-L model. For each pair of three data sets, all the estimates of  $\sigma_{\eta,12}$ , which is the parameter of the instantaneous correlation of volatility, are significant at the five percent level. While the correlation between the variables,  $\rho_{12}$ , is significant for

**TABLE 1** MCL estimates of the univariate SV-L and SV-LSE models

Parameter	SV-L			SV-LSE		
	S&P	Nikkei	Hang Seng	S&P	Nikkei	Hang Seng
$\phi$	0.9606 (0.0083)	0.9635 (0.0064)	0.93564 (0.0094)	0.9821 (0.0052)	0.9819 (0.0054)	0.9799 (0.0061)
$\sigma_\eta$	0.2221 (0.0235)	0.2466 (0.0214)	0.3444 (0.0239)	0.1907 (0.0177)	0.2057 (0.0177)	0.2745 (0.0184)
$\mu$	-0.0177 (0.0054)	0.0037 (0.0039)	0.0294 (0.0069)	0.0331 (0.0095)	0.0332 (0.0093)	0.0647 (0.0101)
$\lambda$	-0.3028 (0.0513)	-0.4376 (0.0396)	-0.2902 (0.0435)			
$\gamma_1$				-0.1149 (0.0159)	-0.0928 (0.0100)	-0.0646 (0.0089)
$\gamma_2$				-0.0615 (0.0128)	-0.0338 (0.0101)	-0.0516 (0.0091)
LogLik	-7876.3	-8281.8	-8107.5	-7846.6	-8088.9	-7971.5
AIC	15760.6	16572.6	16223.2	15703.1	16187.8	15952.9
BIC	15785.4	16596.4	16247.8	15734.1	16218.7	15983.8

LogLik is the log-likelihood based on  $\ln y_t^2$ , and AIC and BIC are calculated based on LogLik.

**TABLE 2** MCL estimates for the bivariate SV-L model

Parameter	S&P	Nikkei	S&P	Hang Seng	Nikkei	Hang Seng
$\phi_i$	0.9638 (0.0074)	0.9701 (0.0050)	0.9606 (0.0090)	0.9536 (0.0096)	0.9627 (0.0052)	0.9478 (0.0064)
$\sigma_{\eta,ii}$	0.0459 (0.0095)	0.0231 (0.0042)	0.0558 (0.0011)	0.0407 (0.0078)	0.0559 (0.0070)	0.0314 (0.0051)
$\sigma_{\eta,12}$	0.0548 (0.0078)		0.0943 (0.0179)		0.0944 (0.0099)	
$\mu_i$	-0.0169 (0.0050)	0.0030 (0.0032)	-0.0180 (0.0063)	0.0220 (0.0068)	0.0045 (0.0034)	0.0245 (0.0056)
$\lambda_i$	-0.2744 (0.0518)	-0.4153 (0.0449)	-0.2743 (0.0263)	-0.2359 (0.0278)	-0.3339 (0.0138)	-0.1984 (0.0138)
$\rho_{12}$	0.1251 (0.0582)		0.0359 (0.0232)		0.3190 (0.0322)	
LogLik	-16144.7		-15956.7		-16369.9	
AIC	32309.4		31922.4		32741.8	
BIC	32371.3		31995.3		32821.7	

LogLik is the log-likelihood based on  $\ln y_t^2$ , and AIC and BIC are calculated based on LogLik.

the pairs (S&P, Nikkei) and (Nikkei, Hang Seng), it is insignificant for the pair (S&P, Hang Seng). The significance (and insignificance) and signs of the estimates of the other parameters are unchanged from the univariate case in Table 1.

For the bivariate SV-LSE model, the MCL estimates in Table 3 indicate that the estimates of  $\sigma_{\eta,12}$  and  $\rho_{12}$  are significant for all three pairs of

**TABLE 3** MCL estimates for the bivariate SV-LSE model

Parameter	S&P	Nikkei	S&P	Hang Seng	Nikkei	Hang Seng
$\phi_i$	0.9882 (0.0038)	0.9845 (0.0047)	0.9821 (0.0059)	0.9799 (0.0065)	0.9819 (0.0054)	0.9799 (0.0071)
$\sigma_{\eta,ii}$	0.0342 (0.0064)	0.0380 (0.0059)	0.0420 (0.0077)	0.0759 (0.0112)	0.0376 (0.0060)	0.0741 (0.0110)
$\sigma_{\eta,12}$	0.0180 (0.0038)		0.0302 (0.0053)		0.0243 (0.0047)	
$\mu_i$	0.0357 (0.0081)	0.0288 (0.0083)	0.0370 (0.0099)	0.0604 (0.0109)	0.0291 (0.0091)	0.0623 (0.0104)
$\gamma_{1i}$	-0.1038 (0.0153)	-0.0905 (0.0093)	-0.1115 (0.0156)	-0.0594 (0.0086)	-0.0855 (0.0093)	-0.0634 (0.0088)
$\gamma_{2i}$	-0.0621 (0.0114)	-0.0297 (0.0090)	-0.0676 (0.0128)	-0.0476 (0.0101)	-0.0291 (0.0100)	-0.0493 (0.0097)
$\rho_{12}$	0.0029 (0.2540)		0.0007 (0.3549)		0.3527 (0.0362)	
LogLik	-15926.3		-15798.3		-16039.4	
AIC	31876.6		31620.7		32102.7	
BIC	31950.9		31694.9		32177.0	

LogLik is the log-likelihood based on  $\ln y_t^2$ , and AIC and BIC are calculated based on LogLik.

variables, which implies the existence of spillovers in both the mean and the volatility equations. Comparing AIC and BIC in Table 3 for SV-LSE with those in Table 2 for SV-L, the bivariate SV-L model is preferred for all data sets. The results in Tables 1–3 also indicate that the likelihood ratio tests for the null hypothesis  $\sigma_{\eta,ij} = \rho_{ij} = 0$  for any  $i, j$  ( $i \neq j$ ) reject the univariate models in favor of their bivariate counterparts.

Table 4 shows the results for the trivariate SV-L model. All the estimated off-diagonal elements of  $\Sigma_{\eta}$  and  $P_{\varepsilon}$  are significant at the five percent level, except for  $\rho_{13}$ , which is the correlation between the conditional distributions of the S&P and the Hang Seng. This result is consistent with the estimates from the bivariate models. The significance of the estimated off-diagonal elements of  $\Sigma_{\eta}$  and  $P_{\varepsilon}$  implies the rejection of the univariate models in favor of the trivariate SV-L model. The signs and significance of the other parameter estimates are unchanged from the univariate case.

Table 5 presents the MCL estimates for the trivariate SV-LSE model. As all the estimated off-diagonal elements of  $\Sigma_{\eta}$  and  $P_{\varepsilon}$  are significant at the five percent level, this result is also consistent with the estimates from the bivariate models. The significance of all the off-diagonal elements of  $\Sigma_{\eta}$  and  $P_{\varepsilon}$  implies the rejection of the univariate models in favor of the trivariate SV-LSE model. The signs and significance of the other estimated parameter are unchanged from the univariate case.

**TABLE 4** MCL estimates for the trivariate SV-L model

Parameter	S&P	Nikkei	Hang Seng
$\phi_i$	0.9612 (0.0072)	0.9636 (0.0057)	0.9361 (0.0109)
$\sigma_{\eta,1i}$	0.0514 (0.0085)	0.0248 (0.0047)	0.0364 (0.0067)
$\sigma_{\eta,2i}$		0.0618 (0.0084)	0.0312 (0.0060)
$\sigma_{\eta,3i}$			0.1187 (0.0181)
$\mu_i$	-0.0173 (0.0046)	0.0035 (0.0039)	0.0297 (0.0073)
$\lambda_i$	-0.2493 (0.0373)	-0.4166 (0.0360)	-0.2799 (0.0334)
$\rho_{1i}$	1	0.1487 (0.0369)	0.0574 (0.0423)
$\rho_{2i}$		1	0.2451 (0.0381)
LogLike	-24210.5		
AIC	48457.0		
BIC	48568.4		

LogLik is the log-likelihood based on  $\ln y_i^2$ , and AIC and BIC are calculated based on LogLik.

**TABLE 5** MCL estimates for the trivariate SV-LSE model

Parameter	S&P	Nikkei	Hang Seng
$\phi_i$	0.9874 (0.0039)	0.9835 (0.0050)	0.9810 (0.0072)
$\sigma_{\eta,1i}$	0.0388 (0.0065)	0.0183 (0.0039)	0.0308 (0.0049)
$\sigma_{\eta,2i}$		0.0378 (0.0058)	0.0255 (0.0049)
$\sigma_{\eta,3i}$			0.0766 (0.0106)
$\mu_i$	0.0383 (0.0081)	0.0284 (0.0084)	0.0599 (0.0098)
$\gamma_{1i}$	-0.1026 (0.0146)	-0.0865 (0.0091)	-0.0599 (0.0084)
$\gamma_{2i}$	-0.0666 (0.0110)	-0.0288 (0.0092)	-0.0478 (0.0092)
$\rho_{1i}$	1	0.0001 (0.4567)	0.0000 (0.1723)
$\rho_{2i}$		1	0.3411 (0.0421)
LogLike	-23852.3		
AIC	47746.5		
BIC	47876.6		

LogLik is the log-likelihood based on  $\ln y_t^2$ , and AIC and BIC are calculated based on LogLik.

## 5. CONCLUDING REMARKS

In this paper, two types of asymmetric multivariate stochastic volatility (SV) models were proposed and analyzed, namely, the SV with leverage (SV-L) model, based on the negative correlation between the innovations in the returns and volatility, and the SV with leverage and size effect (SV-LSE) model, based on the signs and magnitude of the returns.

The paper derived the state space form for the logarithm of the squared returns, which follow the multivariate SV-L model and developed estimation methods for the multivariate SV-L and SV-LSE models based on the Monte Carlo likelihood (MCL) approach. The empirical results showed that the multivariate SV-LSE model fits the data more accurately with respect to AIC and BIC than does the multivariate SV-L model to the bivariate and trivariate returns of the S&P 500, Nikkei 225, and Hang Seng indexes. Moreover, the empirical results suggest that the univariate models should be rejected in favor of their bivariate and trivariate counterparts.

The asymmetric multivariate SV-LSE and SV-L models can be extended in terms of distributional considerations

(1) Modeling the tails of the conditional distribution: A direct way of accommodating this problem is to assume the Student  $t$  distribution

or a mixture of two or more normal distributions (for further details, see Bai et al., 2003; Liesenfeld and Jung, 2000; Watanabe and Asai, 2003, among others). In this context, we may consider asymmetric multivariate SV models with heavy-tailed distributions.

(2) An alternative is to consider the two-factor model analyzed by Chernov et al. (2003). In their specification, the second SV factor is expected to act as a factor dedicated to the exclusive modeling of the tail behavior. The empirical analysis of Asai (2005) indicates that AIC and BIC tend to select two-factors among multifactors for the S&P 500 and the TOPIX stock returns. Based on these results, the asymmetric multivariate and multifactor SV model would seem to be useful candidates for extension.

### APPENDIX A: SOME MOMENTS OF THE FOLDED NORMAL AND HALF NORMAL DISTRIBUTIONS

If  $X$  has a normal distribution with mean  $\mu$  and variance  $\sigma^2$ ,  $X \sim N(\mu, \sigma^2)$ , then  $Y = |X|$  is said to have a *folded normal distribution*. Leone et al. (1961) discussed various properties and applications of the folded normal distribution, while Elandt (1961) derived the general formula for the moments. The first and second moments are given by

$$E(Y) = \sqrt{\frac{2}{\pi}} \sigma \exp\left(-\frac{\mu^2}{2\sigma^2}\right) - \mu \left[1 - 2\Phi\left(\frac{\mu}{\sigma}\right)\right],$$

$$E(Y^2) = \mu^2 + \sigma^2,$$

where  $\Phi(x)$  is the distribution function of the standard normal distribution. If the folding is about the mean, so that  $\mu = 0$ , this leads to a *half-normal distribution*. The first and second moments of the half-normal distribution are  $\sqrt{2/\pi}$  and  $\sigma^2$ , respectively. For odd moments,  $E[Y^{2n+1}] = (2n) \cdots 6 \cdot 4 \cdot 2 \sigma^{2n+1} \sqrt{2/\pi}$  ( $n = 1, 2, 3, \dots$ ).

This paper uses the mean of the folded normal distribution, which requires the derivation of two expectations, namely  $E[\Phi(a|\varepsilon)|\varepsilon^2]$  and  $E[\Phi(a|\varepsilon)|\varepsilon \ln \varepsilon^2]$ , where  $a$  is a constant and  $|\varepsilon|$  follows a half-normal distribution with unit variance.

In order to derive these expectations, it is convenient to use the power series of  $\Phi(z)$ , as given by

$$\Phi(z) = \frac{1}{2} + \phi(z) \sum_{n=0}^{\infty} \frac{z^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)},$$

where  $\phi(z)$  is the standard normal density function (see Abramowitz and Stegun, 1970, Section 26.2.11). Substituting the power series into the above expectations leads to

$$E[\Phi(a|\varepsilon)|\varepsilon^2] = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{a^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)} E[\phi(a|\varepsilon)|\varepsilon|^{2n+2}],$$

$$E[\Phi(a|\varepsilon)|\varepsilon|\ln \varepsilon^2] = \frac{1}{2} E[|\varepsilon|\ln \varepsilon^2] + \sum_{n=0}^{\infty} \frac{a^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \times E[\phi(a|\varepsilon)|\varepsilon|^{2n+2} \ln \varepsilon^2],$$

as it is possible to exchange the summation and integral for these cases. Based on the odd moments of the half-normal distribution, we have

$$E[\Phi(a|\varepsilon)|\varepsilon^2] = \frac{1}{\pi} \sum_{n=0}^{\infty} a^{2n+1} (1+a^2)^{-n-3/2} \frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{1 \cdot 3 \cdot 5 \cdots (2n+1)}.$$

In order to obtain an analytical solution of the above expression, the expectations of the chi-squared distribution and a property of the gamma function are required, namely,

$$E[|\varepsilon|\ln \varepsilon^2] = \sqrt{\frac{2}{\pi}} E[\ln \chi_2],$$

$$E[\chi_v^{n+1} \ln \chi_v] = \frac{2^{n+1} \Gamma(n+1+v/2)}{\Gamma(v/2)} E[\ln \chi_{2n+2+v}],$$

where  $\chi_v$  follows a chi-squared distribution with degrees of freedom given by  $v$ .

According to Abramowitz and Stegun (1970, Section 26.4.36),  $E[\ln \chi_v] = \psi(v/2) + \ln 2$ , where  $\psi(\cdot)$  is the digamma function, which leads to

$$E[\Phi(a|\varepsilon)|\varepsilon|\ln \varepsilon^2] = \frac{1}{\sqrt{2\pi}} \left[ \{\psi(1) + \ln 2\} + a(1+a^2)^{-1/2} + \sum_{n=0}^{\infty} a^{2n+1} (1+a^2)^{-n-3/2} \{\psi(n+3/2) + \ln 2\} \right]$$

**APPENDIX B: MOMENTS OF THE MULTIVARIATE HALF-NORMAL DISTRIBUTION**

In order to consider the multivariate half-normal distribution, consider an  $m$ -dimensional random vector,  $X$ , which follows a multivariate normal

distribution with mean zero and correlation matrix given by  $P = \{\rho_{ij}\}$ . As  $X \sim N(0, P)$ ,  $|X|$  is said to have a multivariate half-normal distribution. It is straightforward to extend this result to a more general covariance matrix. The purpose of this appendix is to derive the first two moments of  $|X|$  and the expectation of the outer product of  $|X|$  and  $\ln X^2$ .

First, consider a cross product of the elements of  $X$ . For  $i \neq j$ , we have

$$E|x_i x_j| = E[|x_i| E(|x_j| | x_i)].$$

Noting that  $x_j | x_i \sim N(\rho_{ij} x_i, 1 - \rho_{ij}^2)$  and  $|x_j| | x_i$  has the folded normal distribution, we obtain the moments of  $|x_i x_j|$  as

$$E|x_i x_j| = \frac{2}{\pi} (1 + a^2)^{-3/2} - \rho_{ij} [1 - 2E[\Phi(a|x_i|) x_i^2]],$$

where  $a = \rho_{ij} / \sqrt{1 - \rho_{ij}^2}$ . Appendix A shows the closed-form solution of the last expectation, and also the first two moments of  $|x_i|$ . Therefore we have the first two moments of the multivariate half-normal distribution as

$$E|X| = \sqrt{\frac{2}{\pi}} t, \quad E|XX'| = P_{|X|},$$

where the  $(i, i)$  element of  $P_{|X|}$  is one, while the  $(i, j)$  element is given by

$$\frac{2}{\pi} (1 - \rho_{ij}^2)^{3/2} \left[ 1 + \sum_{n=0}^{\infty} \rho_{ij}^{2n+2} \frac{2 \cdot 4 \cdot 6 \cdots (2n + 2)}{1 \cdot 3 \cdot 5 \cdots (2n + 1)} \right].$$

Now consider the expectation of the outer product of  $|X|$  and  $\ln X^2$ , that is,  $R_{|X|} \equiv E[|X| \{\ln X^2\}']$ . For the univariate standard normal variable  $x$ , where  $x \sim N(0, 1)$ , Harvey and Shephard (1996) showed that

$$E(|x| \ln |x|^2) = \{\psi(1/2) + \ln 2\} \sqrt{2/\pi}.$$

Thus the  $(i, i)$  element of  $R_{|X|}$  is  $\{\psi(1/2) + \ln 2\} \sqrt{2/\pi}$ , so that it is only necessary to consider the off-diagonal elements of  $R_{|X|}$ ,  $E[|x_i| \ln x_j^2]$ . As the expectation of  $|x_i| | x_j$  is obtained as the mean of the folded normal distribution, it follows that

$$\begin{aligned} E[|x_i| \ln x_j^2] &= E[E[|x_i| | x_j] \ln x_j^2] \\ &= \sqrt{\frac{2}{\pi}} (1 + a^2)^{-1/2} E \left[ \exp\left(-\frac{1}{2} a^2 x_j^2\right) \ln x_j^2 \right] \\ &\quad - \rho_{ij} \left[ \left\{ \psi\left(\frac{1}{2}\right) + \ln 2 \right\} \sqrt{\frac{2}{\pi}} - 2E[\Phi(a|x_j|) |x_j| \ln x_j^2] \right]. \end{aligned}$$

For the right-hand side, we can derive the solution of the first term by tedious calculation, while the last term is given in Appendix A. Therefore, the  $(i, j)$  element of  $R_{|X|}$  is given by

$$\sqrt{\frac{2}{\pi}} \left[ \ln(1 - \rho_{ij}^2) + (1 - \rho_{ij}^2) \sum_{n=0}^{\infty} \rho_{ij}^{2n} \left\{ \psi\left(\frac{2n+1}{2}\right) + \ln 2 \right\} \right].$$

**APPENDIX C: MOMENTS OF THE TRANSFORMED LEVERAGE MSV MODEL**

The logarithmic transformation of the leverage MSV model yields the state spaceform of Equations (10) and (11), which will be developed in this Appendix. Let  $E_s$  denote the expectation conditional on the signs of  $y_t, s_t$ , and assign a similar interpretation to the respective variance and covariance operators. Recalling the fact that  $\eta_t | \varepsilon_t \sim N(LP_\varepsilon^{-1}\varepsilon_t, \Sigma_\eta - LP_\varepsilon^{-1}L)$ , and the results of Appendix B, we have

$$\begin{aligned} \mu_t^* &\equiv E_s(\eta_t) = E_s[E(\eta_t | \varepsilon_t)] = E_s[LP_\varepsilon^{-1}\varepsilon_t] = \sqrt{\frac{2}{\pi}} LP_\varepsilon^{-1} s_t, \\ \Sigma_{\eta,t}^* &\equiv V_s(\eta_t) = E_s(\eta_t \eta_t') - E_s(\eta_t)E_s(\eta_t') \\ &= E_s[E(\eta_t \eta_t' | \varepsilon_t)] - \frac{2}{\pi} LP_\varepsilon^{-1} s_t s_t' P_\varepsilon^{-1} L \\ &= \Sigma_\eta - LP_\varepsilon^{-1} L + LP_\varepsilon^{-1} E_s(\varepsilon_t \varepsilon_t') P_\varepsilon^{-1} L - \frac{2}{\pi} LP_\varepsilon^{-1} s_t s_t' P_\varepsilon^{-1} L \\ &= \Sigma_\eta - LP_\varepsilon^{-1} L + LP_\varepsilon^{-1} \left[ \left\{ P_{|\varepsilon|} - \frac{2}{\pi} W' \right\} \circ (s_t s_t') \right] P_\varepsilon^{-1} L, \end{aligned}$$

and

$$\begin{aligned} L_t^* &\equiv Cov_s(\eta_t, \xi_t) = E_s[\eta_t \{\ln \varepsilon_t^2\}'] - E_s(\eta_t)E(\{\ln \varepsilon_t^2\}') \\ &= E_s[E(\eta_t | \varepsilon_t) \{\ln \varepsilon_t^2\}'] - c \sqrt{\frac{2}{\pi}} LP_\varepsilon^{-1} s_t t' \\ &= LP_\varepsilon^{-1} \left[ E_s[\varepsilon_t \{\ln \varepsilon_t^2\}'] - c \sqrt{\frac{2}{\pi}} s_t t' \right] \\ &= LP_\varepsilon^{-1} \left[ \left\{ R_{|\varepsilon|} - c \sqrt{\frac{2}{\pi}} \right\} \circ (s_t t') \right], \end{aligned}$$

where  $P_{|\varepsilon|}$  and  $R_{|\varepsilon|}$  are the correlation matrices of the multivariate half-normal distribution arising from  $\varepsilon_t \sim N(0, P_\varepsilon)$  and the expectation of the

outer product of  $|\varepsilon_t|$  and  $\ln \varepsilon_t^2$ , respectively. The constant  $c$  is defined by  $c = \psi(1/2) + \ln 2 = -1.2703$ .

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