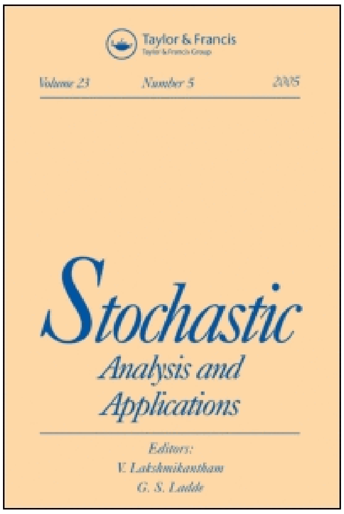

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ON KILLED PROCESSES AND STOPPED FILTRATIONS

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ABSTRACT

This note considers three σ-algebras that contain information about what happens to a stochastic process prior to a random time T. It is shown that the three σ-algebras coincide in a canonical setting but not in general.

1. INTRODUCTION

Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces and let $Z_s : \Omega \to E$ be $\mathcal{F}/\mathcal{E}$-measurable for each $s \in \mathrm{R} \equiv$ the real line. Thus $Z \equiv (Z_s)_{\mathrm{R}}$ is a *process* defined on $(\Omega, \mathcal{F})$ with state space $(E, \mathcal{E})$. [Our treatment here is measure-free so we drop the qualifier "stochastic".] Let $\mathcal{E}^{\mathrm{R}}$ be the product σ-algebra on E^{R} generated by sets of the form $E^{(-\infty,t)} \times C \times E^{(t,\infty)}$, $t \in \mathrm{R}$, $C \in \mathcal{E}$. The process Z can equivalently be regarded as an $\mathcal{F}/\mathcal{E}^{\mathrm{R}}$-measurable mapping from Ω to E^{R}, $\omega \to (Z_s(\omega))_{\mathrm{R}}$. Fix $\delta \in E$ and define $k_t Z$, the process Z killed at time t, by

$$k_t Z_s(\omega) = \begin{Bmatrix} Z_s(\omega) & \text{if } s < t \\ \delta & \text{if } s \geq t \end{Bmatrix}, \quad \omega \in \Omega.$$

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Let $\mathcal{B}$ be the Borel subsets of R, let $T : \Omega \to \mathrm{R}$ be $\mathcal{F}/\mathcal{B}$-measurable and define the process $k_T Z$ by

$$k_T Z_s(\omega) = \begin{Bmatrix} Z_s(\omega) & \text{if } s < T(\omega) \\ \delta & \text{if } s \geq T(\omega) \end{Bmatrix}, \quad \omega \in \Omega.$$

Define a filtration $(\mathcal{F}_t)_\mathrm{R}$ by

$$\mathcal{F}_t = \sigma\{T, k_t Z\} = (T, k_t Z)^{-1} \mathcal{B} \otimes \mathcal{E}^\mathrm{R}$$

and put

$$\mathcal{F}_\infty = \sigma\{T, Z\} = (T, Z)^{-1} \mathcal{B} \otimes \mathcal{E}^\mathrm{R} .$$

Here we are concerned with the following σ-algebras:

$$\sigma\{T, k_T Z\} = (T, k_T Z)^{-1} \mathcal{B} \otimes \mathcal{E}^\mathrm{R},$$

$$\mathcal{F}_{T\leq} = \{A \in \mathcal{F}_\infty : A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall \ t \in \mathrm{R}\},$$

$$\mathcal{F}_{T=} = \{A \in \mathcal{F}_\infty : A \cap \{T = t\} \in \mathcal{F}_t \ \forall \ t \in \mathrm{R}\}.$$

It is readily checked that all three coincide if T takes on countably many values. We shall show that this equality holds within a canonical framework (Section 2) but that the three may differ in noncanonical applications (Section 3).

For more information on stopped filtrations and killed processes, the reader is referred to [1] and [2]. Note that the σ-algebra $\mathcal{F}_{T\leq}$ is usually denoted by $\mathcal{F}_T$. We add "≤" to distinguish between $\mathcal{F}_{T\leq}$ and $\mathcal{F}_{T=}$. Also $\sigma\{T, k_T Z\}$ equals the σ-algebra $\mathcal{F}_{T-} \equiv \sigma\{A \cap \{T > t\}, A \in \mathcal{F}_t, t \in \mathrm{R}\}$; see Propositions 1 and 25 in [1]. Before turning to the canonical case, we establish the following result. The first inclusion follows from Propositions 13 and 25 in [1] however for clarity a self-contained proof is given here.

Proposition 1: $\sigma\{T, k_T Z\} \subseteq \mathcal{F}_{T\leq} \subseteq \mathcal{F}_{T=}$.

Proof: Put $\eta = f(T, k_T Z)$, where $f : \mathrm{R} \times E^{\mathrm{R}} \to \mathrm{R}$ is $\mathcal{B}\otimes\mathcal{E}^{\mathrm{R}}/\mathcal{B}$-measurable. For each $t \in \mathrm{R}$ we have

$$\eta 1_{\{T \le t\}} = f(T, k_T(k_t Z)) 1_{\{T \le t\}} \in \mathcal{F}_t$$

and thus η is $\mathcal{F}_{T\le}/\mathcal{B}$-measurable. This proves the first inclusion.

If $A \in \mathcal{F}_{T\le}$ then $A \cap \{T \le s\} \in \mathcal{F}_t$ for all $s < t$. Taking the union over rational $s < t$ yields $A \cap \{T < t\} \in \mathcal{F}_t$. Thus

$$A \cap \{T = t\} = [A \cap \{T \le t\}] \setminus [A \cap \{T < t\}] \in \mathcal{F}_t,$$

and the second inclusion is established. □

2. THE CANONICAL CASE

Consider the following canonical framework. Let F be any subset of E^{R} which is closed under killing; i.e., $y \in F \Rightarrow k_t y \in F$ for all $t \in \mathrm{R}$. Put

$$\Omega = \mathrm{R} \times F,$$

$$\mathcal{F} = \{A \cap \Omega : A \in \mathcal{B}\otimes\mathcal{E}^{\mathrm{R}}\} = \mathcal{B}\otimes\{Y \cap F : Y \in \mathcal{E}^{\mathrm{R}}\},$$

$$\omega = (x, (y_s)_{\mathrm{R}}) \text{ where } x \in \mathrm{R} \text{ and } (y_s)_{\mathrm{R}} \in F,$$

$$T(\omega) = x,$$

$$Z_s(\omega) = y_s.$$

An obvious example is $F = E^{\mathrm{R}}$ in which case $\mathcal{F} = \mathcal{B}\otimes\mathcal{E}^{\mathrm{R}}$. Another standard example is the following: if E is Polish, $\mathcal{E}$ consists of the Borel subsets of E, and $F \equiv D(\mathrm{R},E)$ is the set of right continuous functions with left-hand limits then $\mathcal{F} = \mathcal{B}\otimes\mathcal{D}$ where $\mathcal{D}$ is the σ-algebra on $D(\mathrm{R},E)$ generated by the Skorohod topology.

Proposition 2: In the canonical case we have $\sigma\{T, k_T Z\} = \mathcal{F}_{T\le} = \mathcal{F}_{T=}$.

Proof: By Proposition 1 it suffices to show that $\mathcal{F}_{T=} \subseteq \sigma\{T, k_T Z\}$. Let $\eta : \Omega \to R$ be an $\mathcal{F}_{T=}/\mathcal{B}$-measurable function and observe that $(T, k_t Z)$ is Ω-valued since (T, Z) is Ω-valued and F is closed under killing. This implies that, for each $t \in R$, there is a function $f_t : \Omega \to R$ such that

$$\begin{aligned} \eta 1_{\{T=t\}} &= f_t(T, k_t Z) \\ &= f_T(T, k_T Z) 1_{\{T=t\}} \, . \end{aligned}$$

Thus we have

$$\eta = f_T(T, k_T Z) \, . \tag{1}$$

Fix $\omega \in \Omega$ and put

$$\phi = (T(\omega), k_T Z(\omega)) \, . \tag{2}$$

We then have $\phi \in \Omega$ and, by (1),

$$\eta(\phi) = f_{T(\phi)}(T(\phi), k_T Z(\phi)) \, . \tag{3}$$

Now $T(\phi) = T(\omega)$ and $k_T Z(\phi) = k_T Z(\omega)$ so it follows from (2) and (3) with ω arbitrary that

$$\eta(T, k_T Z) = f_T(T, k_T Z) \, . \tag{4}$$

Now (1) and (4) yield

$$\eta = \eta(T, k_T Z) \, . \tag{5}$$

Fix $B \in \mathcal{B}$. Since η is $\mathcal{F}_{T=}/\mathcal{B}$-measurable and since $\mathcal{F}_{T=} \subseteq \mathcal{F}$ it follows that η is $\mathcal{F}/\mathcal{B}$-measurable, so there exists $A \in \mathcal{B} \otimes \mathcal{E}^R$ such that $\eta^{-1} B = A \cap \Omega$. From (5) we obtain

$$\{\eta \in B\} = \{\eta(T, k_T Z) \in B\}$$
$$= \{(T, k_T Z) \in \eta^{-1} B\}$$
$$= \{(T, k_T Z) \in A \cap \Omega\}$$
$$= \{(T, k_T Z) \in A\} \in \sigma\{T, k_T Z\},$$

where the final equality is due to the fact that $(T, k_T Z)$ takes values in Ω. Thus η is $\sigma\{T, k_T Z\}/\mathcal{B}$-measurable implying that $\mathcal{F}_{T=} \subseteq \sigma\{T, k_T Z\}$. □

3. COUNTEREXAMPLES

The following examples show that the equalities $\sigma\{T, k_T Z\} = \mathcal{F}_{T\leq}$ and $\mathcal{F}_{T\leq} = \mathcal{F}_{T=}$ can fail in noncanonical applications.

Example 1. Let D be a nonmeasurable subset of $(0, 1)$; i.e., $D \subseteq (0, 1)$ and $D \notin \mathcal{B}(0, 1)$. Define $\Omega = (0, 2)$, $E = \{0, 1\}$, $\mathcal{E} = \mathcal{B}\{0, 1\}$,

$$Z_s(\omega) = \begin{cases} 1 & \text{if } \omega < s \text{ and } \omega \in D, \\ 0 & \text{otherwise,} \end{cases}$$

$$T(\omega) = \begin{cases} \omega & \text{if } \omega \in D, \\ \omega - 1 & \text{if } \omega - 1 \in D^c \cap (0, 1), \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{F} = \mathcal{F}_\infty = \sigma\{T, Z\}$. Choose $\delta = 0$ and observe that $k_T Z \equiv 0$ since $T(\omega) \leq \omega$ for all $\omega \in \Omega$. We thus have $\sigma\{T, k_T Z\} = \sigma\{T\} = T^{-1}\mathcal{B}$. If $D = T^{-1}B$ for some $B \in \mathcal{B}$ then $B \cap (0, 1) = D$, contradicting $D \notin \mathcal{B}$. Thus $D \notin \sigma\{T, k_T Z\}$.

On the other hand we have $D \in \mathcal{F}_\infty$,

$$D \cap \{T < t\} = \{\omega \in \Omega : \omega \in D \text{ and } \omega < t\}$$
$$= \bigcup_{\text{rational } s < t} \{\omega \in \Omega : Z_s(\omega) = 1\} \in \mathcal{F}_t,$$

and

$$D \cap \{T = t\} = \begin{Bmatrix} \{T = t\} & \text{if } t \in D \\ \varnothing & \text{if } t \notin D \end{Bmatrix} \in \mathcal{F}_t,$$

implying that $D \cap \{T \leq t\} \in \mathcal{F}_t$ and thus $D \in \mathcal{F}_{T\leq}$.

<u>Example 2</u>. Again let D be a nonmeasurable subset of $(0,1)$. Define $\Omega = [0, 1]$, $E = \{0, 1\}$, $\mathcal{E} = \mathcal{B}\{0, 1\}$,

$$Z_s(\omega) = \begin{cases} 1 & \text{if } \omega \in D \text{ and } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$T(\omega) = \omega$, and $\mathcal{F} = \mathcal{F}_\infty = \sigma\{T, Z\}$. Note that $D \in \mathcal{F}_\infty$ so $\mathcal{F}_\infty \neq \mathcal{B}[0,1]$. Choose $\delta = 0$ and observe that

$$k_t Z_s(\omega) = \begin{cases} 1 & \text{if } \omega \in D \text{ and } 1 = s < t, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$\mathcal{F}_t = \begin{cases} \mathcal{B}[0,1] & \text{if } t \leq 1, \\ \mathcal{F} & \text{if } t > 1. \end{cases}$$

Thus

$$\mathcal{F}_{T\leq} = \{A \in \mathcal{F}_\infty : A \cap [0,t] \in \mathcal{B}[0,1] \ \forall \ t \leq 1\} = \mathcal{B}[0,1]$$

and

$$\mathcal{F}_{T=} = \{A \in \mathcal{F}_\infty : A \cap \{t\} \in \mathcal{B}[0,1] \ \forall \ t \leq 1\} = \mathcal{F}_\infty .$$

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