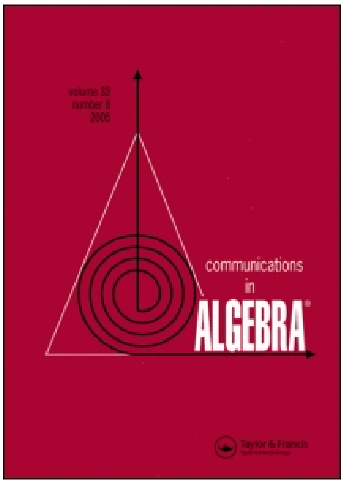

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ON THE DIMENSION OF THE IRREDUCIBLE MODULES FOR SEMISIMPLE HOPF ALGEBRAS

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1 INTRODUCTION

In [4], Kaplansky conjectured that if H is a semisimple Hopf algebra over an algebraically closed field of characteristic 0, then for any irreducible H-module V we have $dim_k V \mid dim_k H$. When H is the group algebra of a finite group this is a well known result due to Frobenius (this is the reason why a Hopf algebra which satisfy Kaplansky's conjecture is called of Frobenius type).

In the last years, there have been obtained some results of this type for specific classes of semisimple Hopf algebras. More exactly in [7] it was proved that if H is semisimple and $dim_k H = p^n$, for a prime number p, then H is of Frobenius type; and in [3] it was showed that any quasitriangular semisimple Hopf algebra is of Frobenius type.

It is known (see [2] or [8]) that if G is a finite group then Frobenius' result may be improved in the following sense: if V is an irreducible G-module then $dim_k V \mid [G : Z(G)]$ (this result is attributed in [2] to Schur).

In this paper we shall consider classes of semisimple Hopf algebras which are closed under tensor product, quotient and are of Frobenius type; then for any class $\mathcal{C}$, of this type, we shall prove that we have the following result: if $H \in \mathcal{C}$ and V is an irreducible H-module then

$dim_k\, V \mid [H : Z_{Hopf}(H)]$, where $Z_{Hopf}(H)$ is an appropriate center for a Hopf algebra.

In particular we shall see that a positive answer to Kaplansky's conjecture would allow us to extend the above mentioned result to any semisimple Hopf algebra over an algebraically closed field of characteristic 0.

2 PRELIMINARIES

In this paper k is an algebraically closed field of characteristic 0, $\otimes$ means $\otimes_k$, we shall use Sweedler's "sigma notation", and all Hopf algebras considered are finite dimensional. If V is an H-module and $h \in H$, we denote by $h_{|V} : V \to V$ the multiplication with h and $tr_V(h)$ will be $tr(h_{|V})$. If H is a semisimple Hopf algebra we shall say that H is of Frobenius type if for any irreducible H-module V we have $dim_k\, V \mid dim_k\, H$. For general results about Hopf algebras we refer to [6] and [10].

Let H and L be two semisimple Hopf algebras, then $H \otimes L$ is a semisimple Hopf algebra with the natural structure given by the tensor product. Moreover a nonzero integral in $H \otimes L$ is given by $\Lambda \otimes \tilde{\Lambda}$ (where Λ is a nonzero integral in H and $\tilde{\Lambda}$ is a nonzero integral in L). It is also easy to see that if $K \subseteq H$ is a normal Hopf subalgebra then H/K^+H is also a semisimple Hopf algebra.

If H is a Hopf algebra and V is a finite dimensional H-module then the character $\chi_V \in H^*$ is defined by $\chi_V(a) = tr_V(a)$ for all $a \in H$. If H is semisimple we have orthogonality relations between the characters of irreducible modules, similar to those for characters for irreducible G-modules, where G is a group. More exactly, if $\Lambda \in H$ is an integral such that $\varepsilon(\Lambda) = 1$ (this always exists because H is semisimple, see [6]), V and W are irreducible H-modules, then:

$$\langle \chi_V, \chi_W \rangle = 1 \quad \text{when } V \simeq W \tag{1}$$

$$\langle \chi_V, \chi_W \rangle = 0 \quad \text{when } V \not\simeq W \tag{2}$$

where $\langle \chi_V, \chi_W \rangle = \sum \chi_V(\Lambda_{(1)})\chi_W(S(\Lambda_{(2)}))$. This result may be found in [5] or [11].

For a finite dimensional Hopf algebra H it was defined in [1] the Hopf center of H, as the maximal central Hopf subalgebra of H. It is also known that in any finite dimensional Hopf algebra there is a unique maximal cocommutative Hopf subalgebra (see for example [9]).

In this paper we shall need a Hopf subalgebra being maximal with respect to both these properties, so we denote by $Z_{Hopf}(H)$ the maximal

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central and cocommutative Hopf subalgebra of H, which of course is just the intersection of the above two Hopf subalgebras. It is not difficult to prove that: $Z_{Hopf}(H) = \{a \in H \mid \varepsilon(h) \sum a_{(2)} \otimes a_{(1)} \otimes a_{(3)} = \varepsilon(h) \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)} = \sum a_{(1)} \otimes h_{(1)} a_{(2)} S(h_{(2)}) \otimes a_{(3)}$ for all $h \in H\}$.

Remark 2.1. $Z_{Hopf}(H)$ is a Hopf subalgebra of H, so by the Nichols-Zoeller theorem (see [6]), H is a free $Z_{Hopf}(H)$-module of finite rank. Now by analogy with groups we denote this rank by $[H : Z_{Hopf}(H)]$.

Proposition 2.2. *Let H and L be two semisimple Hopf algebras. If $U \in H-$mod and $V \in L-$mod are irreducible modules, then $U \otimes V \in (H \otimes L) -$mod is also an irreducible module and $\chi_{U\otimes V}(h \otimes l) = \chi_U(h)\chi_V(l)$ for all $h, l \in H$. Moreover, any irreducible $H \otimes L$-module is of this form.*

Proof: The statements are actually general facts for semisimple algebras over algebraically closed fields, so we shall not give any prove. □

3 THE MAIN RESULT AND EXAMPLES

We first have to give a definition which establishes the type of classes which are of interest in our paper.

Definition 3.1. *Let $\mathcal{C}$ be a class of semisimple Hopf algebras such that:*

i) if $H, L \in \mathcal{C}$ then $H \otimes L \in \mathcal{C}$
ii) if $H \in \mathcal{C}$ and $K \subseteq H$ is a normal Hopf subalgebra then $H/HK^+ \in \mathcal{C}$
iii) for all $H \in \mathcal{C}$, H is of Frobenius type.

Then we shall say that $\mathcal{C}$ is a closed class of Frobenius type.

Example 3.2. Let p be a fixed prime number and $\mathcal{C}_p$ the class of all semisimple Hopf algebras H, with $dim_k H = p^n$ for some $n \in \mathbf{N}^*$; then from [7] it follows that $\mathcal{C}_p$ is a closed class of Frobenius type.

Example 3.3. If we take $\mathcal{C}'$ to be the class of semisimple Hopf algebras which are quasitriangular, then from [3] it follows that $\mathcal{C}'$ is a closed class of Frobenius type.

We are ready now to give the main result of this paper.

Theorem 3.4. *If $\mathcal{C}$ is a closed class of Frobenius type, $H \in \mathcal{C}$ and U is an irreducible H-module then we have:*

$$dim_k U \mid [H : Z_{Hopf}(H)]$$

Proof. Let U be an irreducible H-module and m a positive integer. We denote by H_m the Hopf algebra $H^{\otimes m}$, then by Proposition 2.2 $U^{\otimes m}$ is an irreducible H_m-module.

$Z_{Hopf}(H)$ is a commutative, cocommutative, semisimple Hopf algebra and so we can identify it with the group algebra of a finite commutative group C_H. We have a map $\pi_m : C_H^m \to C_H$ defined by:

$$\pi_m(c_1, c_2, \ldots, c_m) = c_1 c_2 \ldots c_m$$

and is easy to see that π_m is a group morphism (C_H is commutative). Let K_m be the group algebra $k[ker(\pi_m)]$ and identify $k[C_H^m]$ with $(Z_{Hopf}(H))^{\otimes m}$. Then K_m is a normal Hopf subalgebra in $Z_{Hopf}(H)^{\otimes m}$ (and so K_m is normal in H_m) and

$$dim_k\, K_m = (dim_k\, Z_{Hopf}(H))^{m-1} \tag{3}$$

We want now to prove that $U^{\otimes m}$ is an irreducible $H_m/H_m K_m^+$-module. So let $z = \sum_i \alpha_i(z_1^{(i)}, \ldots, z_m^{(i)}) \in K_m$ where $(z_1^{(i)}, \ldots, z_m^{(i)}) \in ker(\pi_m)$; we chose $\eta_j^{(i)} \in k$ such that: $(z_j^{(i)})_{|U} = \eta_j^{(i)} id_U$ (this is possible because $z_j^{(i)} \in Z(H)$, and U is irreducible), then:

$$\begin{aligned}
\varepsilon(z) id_U &= \sum_i \alpha_i 1_{|U} \\
&= \sum_i \alpha_i \pi_m(z_1^{(i)}, \ldots, z_m^{(i)})_{|U} \\
&= \sum_i \alpha_i (z_1^{(i)} \cdots z_m^{(i)})_{|U} \\
&= \sum_i \alpha_i (\eta_1^{(i)} \cdots \eta_m^{(i)}) id_U
\end{aligned}$$

so if $u = u_1 \otimes \cdots \otimes u_m \in U^{\otimes m}$ we have:

$$\begin{aligned}
zu &= \sum_i \alpha_i z_1^{(i)} u_1 \otimes \cdots \otimes z_m^{(i)} u_m \\
&= \sum_i \alpha_i \eta_1^{(i)} u_1 \otimes \cdots \otimes \eta_m^{(i)} u_m \\
&= \sum_i \alpha_i (\eta_1^{(i)} \cdots \eta_m^{(i)})(u_1 \otimes \cdots \otimes u_m) \\
&= \varepsilon(z) u_1 \otimes \cdots \otimes u_m
\end{aligned}$$

and this can be rewritten as:

$$(z - \varepsilon(z)1)U^{\otimes m} = 0$$

for all $z \in K_m$, which means that $U^{\otimes m}$ is a $H_m/H_m K_m^+$-module. It is irreducible because $U^{\otimes m}$ is an irreducible H_m-module.

Now because $\mathcal{C}$ is a closed class of Frobenius type we have:

$$dim_k\, U^{\otimes m} \mid dim_k\, (H_m/H_m K_m^+)$$

which by (3) and Theorem 3.3.1 from [6] becomes:

$$(dim_k\, U)^m \mid (dim_k\, H)^m/(dim_k\, Z_{Hopf}(H))^{m-1}$$

Since m may be any positive integer we obtain:

$$dim_k\, U \mid [H : Z_{Hopf}(H)]$$

which proves our statement. □

Remark 3.5. If Kaplansky's conjecture is true, that is if any semisimple Hopf algebra over an algebraically closed field of characteristic 0 is of Frobenius type, then the class $\mathcal{C}$ of all semisimple Hopf algebras is a closed class of Frobenius type, so our result may be applied to $\mathcal{C}$.

Remark 3.6. If G is a finite group, Schur's result admits the following improvement (Ito's Theorem, see [2]): Let K be any abelian normal subgroup of G, then the degrees of the irreducible representation of G divide $[G : K]$. So it is natural to ask: If H belongs to a closed class of Frobenius type and K is commutative and cocommutative normal Hopf subalgebra of H, are the dimensions of the irreducible H modules divisors of $[H : K]$?

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