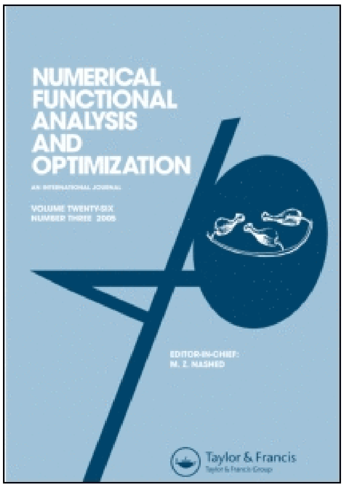

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DISCRETIZATION PRINCIPLES FOR LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS, II

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□ *Consider the boundary value problem* $\mathscr{L}u \equiv -(pu')' + qu' + ru = f,\quad a \le x \le b$, $u(a) = u(b) = 0$. *Let* $H_\nu A_\nu U = \mathbf{f}$ *and* $\widehat{A}_\nu U = \hat{\mathbf{f}}$ *be its finite difference equations and piecewise linear finite element equations on partitions* $\Delta_\nu : a = x_0^\nu < x_1^\nu < \cdots < x_{n_\nu+1}^\nu = b$, $\nu = 1, 2, \ldots$ *with* $h_i^\nu = x_i^\nu - x_{i-1}^\nu$, $h^\nu = \max_i h_i^\nu \to 0$ *as* $\nu \to \infty$, *where* H_ν *are* $n_\nu \times n_\nu$ *diagonal matrices and* A_ν *as well as* $\widehat{A}_\nu$ *are* $n_\nu \times n_\nu$ *tridiagonal. It is shown that the following three conditions are equivalent:* (i) *The boundary value problem has a unique solution* $u \in C^2[a, b]$. (ii) *For sufficiently large* $\nu \ge \nu_0$, *the inverse* $A_\nu^{-1} = (g_{ij}^\nu)$ *exists and* $|g_{ij}^\nu| \le M$, $\forall i, j$ *with a constant* $M > 0$ *independent of* h^ν. (iii) *For sufficiently large* $\nu \ge \hat{\nu}_0$, $\widehat{A}_\nu^{-1} = (\hat{g}_{ij}^\nu)$ *exists and* $|\hat{g}_{ij}^\nu| \le \widehat{M}$, $\forall i, j$ *with a constant* $\widehat{M} > 0$ *independent of* h^ν. *It is also shown by a numerical example that the finite difference method with uniform nodes* $x_{i+1} = x_i + h$, $0 \le i \le n$, $h = (b - a)/(n + 1)$ *applied to the boundary value problem with no solution gives a ghost solution for every* n.

Keywords Discretization principles; Finite difference methods; Finite element methods; Two-point boundary value problems.

AMS Subject Classification 65L10; 65L12; 65L60.

1. INTRODUCTION

According to Carasso [1], we consider the non-self-adjoint boundary value problem

$$\begin{cases} \mathscr{L}u \equiv -\dfrac{d}{dx}\left(p(x)\dfrac{du}{dx}\right) + q(x)\dfrac{du}{dx} + r(x)u = f(x), \quad a \le x \le b, \\ u \in \mathscr{D} = \{u \in C^2[a, b] \mid u(a) = u(b) = 0\} \end{cases} \tag{1.1}$$

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where $p(x) \in C^1[a,b]$, $p(x) \geq p_* > 0$ with a constant p_*, $q(x), r(x), f(x) \in C[a,b]$, but the sign of $r(x)$ may be indefinite.

Then the standard finite difference method (FDM) for solving (1.1) on not necessarily uniform nodes

$$\Delta : a = x_0 < x_1 < \cdots < x_{n+1} = b, \quad h_i = x_i - x_{i-1}, \;\; h = \max_i h_i \tag{1.2}$$

is

$$\begin{cases} L_h U_i = -\dfrac{1}{\omega_i}\left[p_{i+\frac{1}{2}}\dfrac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}}\dfrac{U_i - U_{i-1}}{h_i}\right] \\ \qquad\qquad + \dfrac{q_i}{2\omega_i}(U_{i+1}-U_{i-1}) + r_i U_i = f_i, \qquad 1 \leq i \leq n, \\ U_0 = U_{n+1} = 0, \end{cases} \tag{1.3}$$

where $\omega_i = \frac{1}{2}(h_i + h_{i+1})$, $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$, $p_{i+\frac{1}{2}} = p\big(x_{i+\frac{1}{2}}\big)$, $q_i = q(x_i)$, $r_i = r(x_i)$, $f_i = f(x_i)$ and U_i denote approximations of exact values $u_i = u(x_i)$, $1 \leq i \leq n$.

The equations (1.3) are written in a matrix-vector form

$$HA\mathbf{U} = \mathbf{f}, \tag{1.4}$$

where $H = \mathrm{diag}\big(\omega_1^{-1}, \ldots, \omega_n^{-1}\big)$,

$$A = \begin{pmatrix} \beta_1 & \gamma_1 & & & \\ \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\ & & & \alpha_n & \beta_n \end{pmatrix}$$

with $\alpha_i = -\frac{1}{h_i}p_{i-\frac{1}{2}} - \frac{1}{2}q_i$ $(2 \leq i \leq n)$, $\beta_i = \frac{1}{h_i}p_{i-\frac{1}{2}} + \frac{1}{h_{i+1}}p_{i+\frac{1}{2}} + r_i\omega_i$ $(1 \leq i \leq n)$, $\gamma_i = -\frac{1}{h_{i+1}}p_{i+\frac{1}{2}} + \frac{1}{2}q_i$ $(1 \leq i \leq n-1)$, $\mathbf{U} = (U_1, \ldots, U_n)^t$ and $\mathbf{f} = (f_1, \ldots, f_n)^t$. The matrix A is called a finite difference matrix.

On the other hand, the Galerkin finite element method (FEM), which uses piecewise linear functions

$$\ell_i(x) = \begin{cases} \dfrac{1}{h_i}(x - x_{i-1}) & (x_{i-1} \leq x \leq x_i), \\ \dfrac{1}{h_{i+1}}(x_{i+1} - x) & (x_i \leq x \leq x_{i+1}), \\ 0 & (\text{otherwise}), \quad 1 \leq i \leq n \end{cases}$$

on the nodes (1.2), solves a system of linear equations

$$\widehat{A}\widehat{\mathbf{U}} = \hat{\mathbf{f}}, \tag{1.5}$$

where

$$\widehat{A} = \begin{pmatrix} \hat{\beta}_1 & \hat{\gamma}_1 & & & \\ \hat{\alpha}_2 & \hat{\beta}_2 & \hat{\gamma}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{\alpha}_{n-1} & \hat{\beta}_{n-1} & \hat{\gamma}_{n-1} \\ & & & \hat{\alpha}_n & \hat{\beta}_n \end{pmatrix}$$

with $\hat{\alpha}_i = [\ell_{i-1}, \ell_i]$, $\hat{\beta}_i = [\ell_i, \ell_i]$, $\hat{\gamma}_i = [\ell_{i+1}, \ell_i]$,

$$[\ell_j, \ell_i] = \int_a^b (p\ell_j'\ell_i' + q\ell_j'\ell_i + r\ell_j\ell_i)\,dx, \quad 1 \le i,\ j \le n,$$

$$\widehat{\mathbf{U}} = (\widehat{U}_1, \ldots, \widehat{U}_n)^t \quad \text{and} \quad \hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)^t,$$

$$\hat{f}_i = (f, \ell_i) = \int_a^b f(x)\ell_i(x)\,dx, \quad 1 \le i \le n.$$

The solution $\widehat{\mathbf{U}}$ of (1.5) determines the piecewise linear Galerkin finite element solution $\hat{u}_h(x) = \sum_{j=1}^n \widehat{U}_j\ell_j(x)$, and the matrix $\widehat{A}$ is called a finite element matrix.

In a previous paper [5], the first author of this paper proved the following two theorems:

Theorem 1.1 (A Discretization Principle for FDM). *Let the problem* (1.1) *have a unique solution* $u \in C^2[a, b]$. *Then:*

(i) *The finite difference matrix* A *is nonsingular for sufficiently small* $h > 0$, *that is, for any* $h \le h_0$ *with an appropriate constant* $h_0 > 0$, *so that the linear system* (1.4) *has a unique solution* $\mathbf{U}$ *for any* $h \le h_0$.

(ii) *Let* $h \le h_0$ *and* $A^{-1} = \left(G_{ij}^h\right)$. *Then there exist positive constants* M *and* C_1 *independent of* h *such that*

$$\left|G_{ij}^h\right| \le M, \quad \left|G_{i+1,j}^h - G_{ij}^h\right| \le Mh_{i+1}, \quad \left|G_{i,j+1}^h - G_{ij}^h\right| \le Mh_{j+1}, \ \forall i, j$$

and

$$\max_{i,j}\left|G_{ij}^h - G(x_i, x_j)\right| \le \begin{cases} o(1) & (u \in C^2[a, b]), \\ C_1 h = O(h) & (u \in C^{2,1}[a, b]), \end{cases}$$

where $G(x, \xi)$ *denotes the Green function for* $(\mathscr{L}, \mathscr{D})$ *and* $o(1)$ *means* $o(1) \to 0$ *as* $h \to 0$ *and we observe that* $u \in C^{2,1}[a, b]$ *if* $p \in C^{1,1}[a, b], q, r, f \in C^{0,1}[a, b]$.

(iii) *The error estimates*

$$\max_i |u_i - U_i| \leq \begin{cases} o(1) & (u \in C^2[a, b]), \\ C_2 h = O(h) & (u \in C^{2,1}[a, b], p \in C^{1,1}[a, b], q, r, f \in C^{0,1}[a, b]), \\ C_3 h^2 = O(h^2) & (u \in C^{3,1}[a, b], p \in C^{2,1}[a, b], q, r, f \in C^{1,1}[a, b]) \end{cases}$$

hold, where C_2 *and* C_3 *are appropriate positive constants independent of* h.

Theorem 1.2 (A Discretization Principle for FEM). *Assume that* (1.1) *has a unique solution* $u \in C^{2,1}[a, b]$. *Then:*

(i) *The finite element matrix* $\widehat{A}$ *is nonsingular for sufficiently small* $h > 0$, *that is, for any* $h \leq \hat{h}_0$ *with an appropriate constant* $\hat{h}_0 > 0$.

(ii) *Let* $h \leq \hat{h}_0$ *and* $\widehat{A}^{-1} = \left(\widehat{G}_{ij}^h\right)$. *Then there exist positive constants* $\widehat{M}$ *and* $\widehat{C}_1$ *independent of* h *such that*

$$\left|\widehat{G}_{ij}^h\right| \leq \widehat{M}, \quad \left|\widehat{G}_{i+1,j}^h - \widehat{G}_{ij}^h\right| \leq \widehat{M} h_{i+1}, \quad \left|\widehat{G}_{i,j+1}^h - \widehat{G}_{ij}^h\right| \leq \widehat{M} h_{j+1}, \ \forall i, j$$

and

$$\max_{i,j} \left|\widehat{G}_{ij}^h - G(x_i, x_j)\right| \leq \widehat{C}_1 h = O(h). \tag{1.6}$$

(iii) *Error estimate*

$$\max_i |u_i - \widehat{U}_i| \leq \widehat{C}_3 h^2 = O(h^2) \tag{1.7}$$

holds, where $\widehat{C}_3$ *is a positive constant independent of* h.

Remark 1.1. An inspection of the proof of Theorem 1.2 in [5] shows that, if $u \in C^2[a, b]$, then Theorem 1.2 holds by replacing (1.6) and (1.7) by

$$\max_{i,j} \left|\widehat{G}_{ij}^h - G(x_i, x_j)\right| = o(1) \tag{1.6′}$$

and

$$\max_i |u_i - \widehat{U}_i| \leq \widehat{C}_2 h = O(h), \tag{1.7′}$$

repectively, where $\widehat{C}_2$ is a positive constant independent of h.

In this paper, we shall prove a variant of the above discretization principles. To state it, consider an infinite sequence of partitions

$$\Delta_\nu : a = x_0^\nu < x_1^\nu < \cdots < x_{n_\nu+1}^\nu = b, \quad \nu = 1, 2, 3, \ldots \tag{1.8}$$

such that $h_i^\nu = x_i^\nu - x_{i-1}^\nu$, $h^\nu = \max_i h_i^\nu \to 0$ as $\nu \to \infty$. We denote by A_ν and $\widehat{A}_\nu$ the corresponding finite difference and finite element matrices. Then the following result holds.

Theorem 1.3. *The following three conditions are equivalent.*

(i) *The problem* (1.1) *has a unique solution* $u \in C^2[a, b]$.
(ii) *The matrix* A_ν *is nonsingular for sufficiently large* ν, *that is, for any* $\nu \geq \nu_0$ *with an appropriate positive integer* ν_0. *Let* $A_\nu^{-1} = (g_{ij}^\nu)$, $\nu \geq \nu_0$. (*We use the notation* g_{ij}^ν *in place of* $G_{ij}^{h^\nu}$.) *Then there exists a constant* $M > 0$ *independent of* h^ν *such that* $|g_{ij}^\nu| \leq M$, $\forall i, j$.
(iii) *The matrix* $\widehat{A}_\nu$ *is nonsingular for any* $\nu \geq \hat{\nu}_0$, *where* $\hat{\nu}_0$ *is an appropriate positive integer. Let* $\nu \geq \hat{\nu}_0$ *and* $\widehat{A}_\nu^{-1} = (\hat{g}_{ij}^\nu)$. *Then there exists a constant* $\widehat{M} > 0$ *independent of* h^ν *such that* $|\hat{g}_{ij}^\nu| \leq \widehat{M}$, $\forall i, j$.

A proof of this theorem will be given in Section 2. Furthermore, in Section 3, it will be shown by a numerical example that the uniformly bounded conditions $|g_{ij}^\nu| \leq M$, $\forall i, j$ and $|\hat{g}_{ij}^\nu| \leq \hat{M}$, $\forall i, j$ are necessary in (ii) and (iii) of Theorem 1.3. It is also shown that finite difference and finite element solutions on equidistant nodes for this example give ghost solutions.

2. PROOF OF THEOREM 1.4

In the following, for $u \in C[a, b]$, we use the notation $\|u\|_{[a,b]} = \max_{a \leq x \leq b} |u(x)|$.

In order to prove the equivalence of (i) and (ii), it suffices to show that (ii) implies (i) as Theorem 1.1 is already known. We first remark that the problem (1.1) has a unique solution $u \in C^2[a, b]$ if and only if the map $\mathscr{L} : \mathscr{D} \to C[a, b]$ is injective. Hence suppose, on the contrary, that $\mathscr{L}$ is not injective under the condition (ii). Then, $(\mathscr{L}, \mathscr{D})$ has a zero eigenvalue and its corresponding eigenfunction $v = v(x)$. We then have $\mathscr{L}v = 0$, $v(a) = v(b) = 0$ and $v'(a)v'(b) \neq 0$. (In fact, if $v'(a)v'(b) = 0$, then $\mathscr{L}v = 0$ and $v(a) = v'(a) = 0$ or $v(b) = v'(b) = 0$. This implies $v = 0$ in $[a, b]$, which contradicts that v is an eigenfunction.) Hence the functions $\varphi(x) = \frac{1}{v'(a)} v(x)$ and $\psi(x) = \frac{-1}{v'(b)} v(x)$ are solutions of the initial value problems

$$\mathscr{L}u = 0 \ \ (a \leq x \leq b), \quad u(a) = 0, \quad u'(a) = 1 \tag{2.1}$$

and

$$\mathscr{L}u = 0 \ \ (a \le x \le b), \quad u(b) = 0, \quad u'(b) = -1, \tag{2.2}$$

respectively.

Let $x^{\nu}_{i+\frac{1}{2}} = \frac{1}{2}(x^{\nu}_i + x^{\nu}_{i+1})$, $\alpha^{\nu}_i = -\frac{1}{h^{\nu}_i}p\big(x^{\nu}_{i-\frac{1}{2}}\big) - \frac{1}{2}q(x^{\nu}_i)$, $\beta^{\nu}_i = \frac{1}{h^{\nu}_i}p\big(x^{\nu}_{i-\frac{1}{2}}\big) + \frac{1}{h^{\nu}_{i+1}}p\big(x^{\nu}_{i+\frac{1}{2}}\big) + r(x^{\nu}_i)\omega^{\nu}_i$, $\gamma^{\nu}_i = -\frac{1}{h^{\nu}_{i+1}}p\big(x^{\nu}_{i+\frac{1}{2}}\big) + \frac{1}{2}q(x^{\nu}_i)$, $1 \le i \le n_\nu$, $\nu = 1, 2, \ldots,$ where $\omega^{\nu}_i = (h^{\nu}_i + h^{\nu}_{i+1})/2$. Then the finite difference matrices A_ν corresponding with the partitions (1.8) are

$$A_\nu = \begin{pmatrix} \beta^{\nu}_1 & \gamma^{\nu}_1 & & & \\ \alpha^{\nu}_2 & \beta^{\nu}_2 & \gamma^{\nu}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha^{\nu}_{n_\nu-1} & \beta^{\nu}_{n_\nu-1} & \gamma^{\nu}_{n_\nu-1} \\ & & & \alpha^{\nu}_{n_\nu} & \beta^{\nu}_{n_\nu} \end{pmatrix}, \quad \nu = 1, 2, \ldots.$$

Define two sequences $\{\Phi^{\nu}_i\}$ and $\{\Psi^{\nu}_i\}$ by

$$\begin{aligned} &\Phi^{\nu}_0 = 0, \quad \Phi^{\nu}_1 = h^{\nu}_1, \\ &\Phi^{\nu}_i = -\frac{1}{\gamma^{\nu}_{i-1}}\big(\alpha^{\nu}_{i-1}\Phi^{\nu}_{i-2} + \beta^{\nu}_{i-1}\Phi^{\nu}_{i-1}\big), \quad i = 2, 3, \ldots, n_\nu + 1 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\Psi^{\nu}_{n_\nu+1} = 0, \quad \Psi^{\nu}_{n_\nu} = h^{\nu}_{n_\nu+1}, \\ &\Psi^{\nu}_i = -\frac{1}{\alpha^{\nu}_{i+1}}\big(\beta^{\nu}_{i+1}\Psi^{\nu}_{i+1} + \gamma^{\nu}_{i+1}\Psi^{\nu}_{i+2}\big), \quad i = n_\nu - 1, n_\nu - 2, \ldots, 0, \end{aligned} \tag{2.4}$$

respectively. Let $\sigma^{\nu}_i = L_{h^\nu}\varphi(x^{\nu}_i) = L_{h^\nu}\varphi(x^{\nu}_i) - \mathscr{L}\varphi(x^{\nu}_i) = L^{\nu}_h\big(\varphi(x^{\nu}_i) - \Phi^{\nu}_i\big)$, $\tau^{\nu}_i = L_{h^\nu}\psi(x^{\nu}_i) = L_{h^\nu}\psi(x^{\nu}_i) - \mathscr{L}\psi(x^{\nu}_i) = L^{\nu}_h\big(\psi(x^{\nu}_i) - \Psi^{\nu}_i\big)$, $\boldsymbol{\sigma}^{\nu} = (\sigma^{\nu}_1, \ldots, \sigma^{\nu}_{n_\nu})^t$ and $\boldsymbol{\tau}^{\nu} = (\tau^{\nu}_1, \ldots, \tau^{\nu}_{n_\nu})^t$. Then, for sufficiently small h^ν,

$$\|\boldsymbol{\sigma}^{\nu}\|_\infty \le \begin{cases} o(1) & (\varphi \in C^2[a, b]) \\ C_4 h^\nu = O(h^\nu) & (\varphi \in C^{2,1}[a, b]), \end{cases}$$

$$\|\boldsymbol{\tau}^{\nu}\|_\infty \le \begin{cases} o(1) & (\psi \in C^2[a, b]) \\ C_5 h^\nu = O(h^\nu) & (\psi \in C^{2,1}[a, b]), \end{cases}$$

where C_4 and C_5 are appropriate positive constants independent of h^ν.

We then have from the proof of Lemma 2.1 in [5] that

$$\max_i |\Phi_i^v - \varphi(x_i^v)| \leq \kappa,$$
$$\max_i \left| \frac{\Phi_{i+1}^v - \Phi_i^v}{h_{i+1}^v} - \varphi'(x_i^v) \right| \leq \frac{1}{p_*}\kappa + \frac{1}{2} h^v \|\varphi''\|_{[a,b]} \tag{2.5}$$

and

$$\max_i |\Psi_i^v - \psi(x_i^v)| \leq \tilde{\kappa},$$
$$\max_i \left| \frac{\Psi_{i+1}^v - \Psi_i^v}{h_{i+1}^v} - \psi'(x_i^v) \right| \leq \frac{1}{p_*}\tilde{\kappa} + \frac{1}{2} h^v \|\psi''\|_{[a,b]}, \tag{2.6}$$

where

$$\kappa \leq C_6 h^v + C_7 \|\boldsymbol{\sigma}^v\|_\infty \leq \begin{cases} o(1) & (\varphi \in C^2[a,b]) \\ (C_6 + C_7 \cdot C_4) h^v = O(h^v) & (\varphi \in C^{2,1}[a,b]) \end{cases}$$

and

$$\tilde{\kappa} \leq C_8 h^v + C_9 \|\boldsymbol{\tau}^v\|_\infty \leq \begin{cases} o(1) & (\psi \in C^2[a,b]) \\ (C_8 + C_9 \cdot C_5) h^v = O(h^v) & (\psi \in C^{2,1}[a,b]) \end{cases}$$

with appropriate positive constants C_6–C_9.

Furthermore, from the proof of Lemma 3.1 in [5], we have for $v \geq v_0$

$$g_{ij}^v = \begin{cases} \dfrac{1}{\widetilde{W}_j^v} \Phi_i^v \Psi_j^v & (i \leq j), \\ \dfrac{1}{\widetilde{W}_j^v} \Phi_j^v \Psi_i^v & (i \geq j), \end{cases} \tag{2.7}$$

where

$$\widetilde{W}_j^v \equiv \alpha_j^v \begin{vmatrix} \Phi_{j-1}^v & \Psi_{j-1}^v \\ \Phi_j^v & \Psi_j^v \end{vmatrix} = -\left(p\left(x_{j-\frac{1}{2}}^v\right) + \frac{1}{2} q\left(x_j^v\right) h_j^v \right) \begin{vmatrix} \Phi_{j-1}^v & \Psi_{j-1}^v \\ \frac{\Phi_j^v - \Phi_{j-1}^v}{h_j^v} & \frac{\Psi_j^v - \Psi_{j-1}^v}{h_j^v} \end{vmatrix} \tag{2.8}$$

It now follows from (2.5)–(2.8) that, putting $\varepsilon_v = \max(h^v, \|\boldsymbol{\sigma}^v\|_\infty, \|\boldsymbol{\tau}^v\|_\infty)$, we have

$$\begin{aligned} \widetilde{W}_j^v &= -p\left(x_j^v\right) \begin{vmatrix} \varphi(x_j^v) & \psi(x_j^v) \\ \varphi'(x_j^v) & \psi'(x_j^v) \end{vmatrix} + O(h^v) + O(\|\boldsymbol{\sigma}^v\|_\infty) + O(\|\boldsymbol{\tau}^v\|_\infty) \\ &= O(\varepsilon_v) \to 0 \end{aligned} \tag{2.9}$$

as $\nu \to \infty$, because the determinant vanishes. Let

$$\max_{1 \le i \le n_\nu} |v(x_i^\nu)| = |v(x_{k^\nu}^\nu)|.$$

Then we have

$$|v(x_{k^\nu}^\nu)| \ge \frac{1}{2}\|v\|_{[a,b]} > 0$$

for sufficiently large ν. Hence

$$\begin{aligned}
|g_{k^\nu k^\nu}^\nu| &= \left|\frac{\Phi_{k^\nu}^\nu \Psi_{k^\nu}^\nu}{\widetilde{W}_{k^\nu}^\nu}\right| \\
&= \frac{|\varphi(x_{k^\nu}^\nu) + O(h^\nu) + O(\|\boldsymbol{\sigma}^\nu\|_\infty)|\,|\psi(x_{k^\nu}^\nu) + O(h^\nu) + O(\|\boldsymbol{\tau}^\nu\|_\infty)|}{|\widetilde{W}_{k^\nu}^\nu|} \\
&= \frac{1}{|\widetilde{W}_{k^\nu}^\nu|}\left\{\frac{v(x_{k^\nu}^\nu)^2}{|v'(a)v'(b)|} + O(\varepsilon_\nu)\right\} \\
&\ge \frac{1}{|\widetilde{W}_{k^\nu}^\nu|}\left\{\frac{1}{|v'(a)v'(b)|}\left(\frac{1}{2}\|v\|_{[a,b]}\right)^2 + O(\varepsilon_\nu)\right\} \to \infty
\end{aligned}$$

as $\nu \to \infty$. This contradicts the assumption that $|g_{ij}^\nu| \le M \forall i,j$ for $\nu \ge \nu_0$ with a positive constant M independent of h^ν. We thus conclude that $(\mathscr{L}, \mathscr{D})$ is injective and the problem (1.1) has a unique solution $u \in C^2[a,b]$. This completes the proof of the equivalence of (i) and (ii). Similarly, we can prove the equivalence of (i) and (iii) (cf. Remark 1.1). This proves the theorem.

Corollary 2.1. *Consider the uniform nodes*

$$x_i = a + ih, \quad i = 0,1,2,\ldots,n+1, \quad h = \frac{b-a}{n+1}$$

and the corresponding finite difference and finite element matrices A and $\widehat{A}$. Then the following three conditions are equivalent.

(i) *The problem* (1.1) *has a unique solution $u \in C^2[a,b]$.*
(ii) *A is nonsingular for sufficiently small $h \le h_0$. Let $h \le h_0$ and $A^{-1} = (G_{ij}^h)$. Then there exists a constant $M > 0$ independent of h such that $|G_{ij}^h| \le M$ $\forall i,j$.*
(iii) *$\widehat{A}$ is nonsingular for sufficiently small $h \le \hat{h}_0$. Let $h \le \hat{h}_0$ and $\widehat{A}^{-1} = (\widehat{G}_{ij}^h)$. Then there exists a constant $\widehat{M} > 0$ independent of $h > 0$ such that $|\widehat{G}_{ij}^h| \le \widehat{M}$ $\forall i,j$.*

3. A NUMERICAL EXAMPLE

To illustrate our results, we consider the following problem

$$\begin{cases} \mathscr{L}u \equiv -\dfrac{d^2u}{dx^2} - u = -1, & 0 \le x \le \pi, \\ u \in \mathscr{D} = \{u \in C^2[0,\pi] \mid u(0) = u(\pi) = 0\}, \end{cases} \tag{3.1}$$

which has no solution in $\mathscr{D}$. Then the finite difference matrix A corresponding with the uniform nodes $x_i = ih$, $0 \le i \le n+1$, $h = \frac{\pi}{n+1}$ is

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} - hI \quad (n \times n \text{ matrix}), \tag{3.2}$$

where I denotes the $n \times n$ identity. The eigenvalues λ_j, $j = 1, 2, \ldots, n$ of A are given by

$$\lambda_j = \frac{4}{h}\sin^2\frac{j\pi}{2(n+1)} - h = h\left(\frac{2}{h}\sin\frac{jh}{2} + 1\right)\left(\frac{2}{h}\sin\frac{jh}{2} - 1\right).$$

We have $\lambda_j \neq 0\ \forall j$, because

$$\frac{2}{h}\sin\frac{jh}{2} + 1 > 0 \quad \forall j \ge 1, \qquad \frac{2}{h}\sin\frac{h}{2} - 1 < 0$$

and for $j \ge 2$

$$\begin{aligned} \frac{2}{h}\sin\frac{jh}{2} - 1 &\ge \frac{2}{h}\sin\frac{2h}{2} - 1 \\ &= \frac{2}{h}\left\{h - \frac{1}{6}(h\theta)^3\right\} - 1 \quad (0 < \theta < 1) \\ &> 1 - \frac{1}{3}h^2 \ge 1 - \frac{1}{3}\left(\frac{\pi}{2}\right)^2 > 0. \end{aligned}$$

Hence the $n \times n$ matrix A is nonsingular for any $n \ge 1$. Similarly, we can also prove that the finite element matrix $\widehat{A}$ is nonsingular for any $n \ge 1$. Therefore, by Corollary 2.1, $\max_{i,j}|G_{ij}^h|$ and $\max_{i,j}|\widehat{G}_{ij}^h|$ would diverge as $n \to \infty$ or $h = \frac{\pi}{n+1} \to 0$.

Furthermore, the piecewise linear polynomial $U_h(x) = \sum_{i=1}^n U_i\ell_i(x)$, where $\mathbf{U} = (U_1, \ldots, U_n)^t$ is the solution of the linear system (1.4) with the matrix A defined by (3.2), and the vector $\mathbf{f} = (-1, -1, \ldots, -1)^t \in \mathbb{R}^n$ is

TABLE 1 Behavior of $\max_{i,j}|G_{ij}^h|$, $\max_{i,j}|\widehat{G}_{ij}^h|$, and $\|\mathbf{U}\|_\infty$.

n	$h = \frac{\pi}{n+1}$	$\max_{i,j}\left\|G_{ij}^h\right\|$	$\max_{i,j}\left\|\widehat{G}_{ij}^h\right\|$	$\|\mathbf{U}\|_\infty$
10	2.8560e − 01	9.1987e + 01	9.2932e + 01	1.8469e + 02
100	3.1105e − 02	7.8942e + 03	7.8952e + 03	1.5789e + 04
1000	3.1385e − 03	7.7558e + 05	7.7559e + 05	1.5512e + 06
5000	6.2819e − 04	1.9236e + 07	1.9432e + 07	3.8468e + 07

approximately of the form $c_h \sin x$, where c_h is a constant such that $|c_h| \to \infty$ as $h \to 0$. To prove this, we first remark that in this example, $u = \sin x$ is the solution of both initial value problems (2.1) and (2.2) so that in the proof of Theorem 1.3, we have $\varphi(x) = \psi(x) = \sin x \in C^\infty[0, \pi]$ and $\|\boldsymbol{\sigma}^\nu\|_\infty = \|\boldsymbol{\tau}^\nu\|_\infty = O(h^2)$ as nodes are uniform. Furthermore, writing Φ_i, Ψ_i in place of Φ_i^ν, Ψ_i^ν in (2.5) and (2.6), we have

$$\Phi_1 - \varphi(x_1) = h - \sin h = O(h^3)$$

and

$$\Psi_n - \psi(x_n) = h - \sin nh = h - \sin h = O(h^3)$$

so that κ and $\tilde{\kappa}$ in (2.5) and (2.6) can be replaced by $\kappa = O(h^2) + O(\|\boldsymbol{\sigma}\|_\infty) = O(h^2)$ and $\tilde{\kappa} = O(h^2) + O(\|\boldsymbol{\tau}\|_\infty) = O(h^2)$, respectively (cf. the proof of Lemma 2.1 in [5]). It now follows from (2.5)–(2.7) that

$$G_{ij}^h = \frac{1}{\widetilde{W}_1}(\sin x_i + O(h^2))(\sin x_j + O(h^2)) \quad \forall i, j$$

because $\widetilde{W}_j = \widetilde{W}_1 \, \forall j$ (cf. the proof of Lemma 3.1 in [5]). Hence

$$\begin{aligned} U_i = h \sum_{j=1}^n G_{ij}^h f_j &= -\frac{h}{\widetilde{W}_1}(\sin x_i + O(h^2)) \sum_{j=1}^n (\sin x_j + O(h^2)) \\ &= c_h(\sin x_i + O(h^2)) \fallingdotseq c_h \sin x_i, \end{aligned}$$

where

$$\begin{aligned} c_h &= -\frac{h}{\widetilde{W}_1} \sum_{j=1}^n (\sin x_j + O(h^2)) \\ &= -\frac{h}{\widetilde{W}_1}\left\{\frac{\sin \frac{nh}{2}}{\sin \frac{h}{2}} \sin \frac{(n+1)h}{2} + O(h)\right\} \\ &= -\frac{2}{\widetilde{W}_1}\left(\frac{\frac{h}{2}}{\sin \frac{h}{2}} \cos \frac{h}{2} + O(h^2)\right) = O(h^{-2}), \end{aligned} \tag{3.3}$$

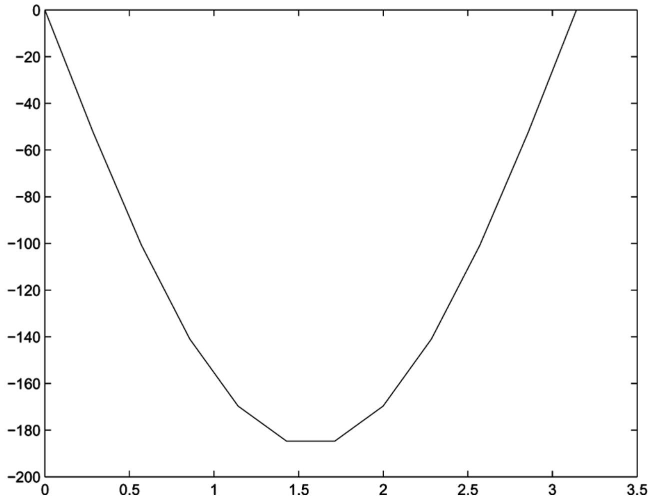

FIGURE 1 $U_h(x)$ $(n = 10, h = \pi/11)$.

because we have $\widetilde{W}_1 = -\frac{1}{h}(\Phi_0\Psi_1 - \Phi_1\Psi_0) = \Psi_0 = \psi(0) + O(h^2) = O(h^2)$. We thus obtain $|c_h| = O(h^{-2}) \to \infty$ as $h \to 0$.

The values of $\max_{i,j}|G_{ij}^h|$, $\max_{i,j}|\widehat{G}_{ij}^h|$ and $\|\mathbf{U}\|_\infty$ for $n = 10, 100, 1000, 5000$ are shown in Table 1.

Graphs of $U_h(x)$ for $n = 10$ and 1000 are shown in Figures 1 and 2, which may be considered ghost solutions.

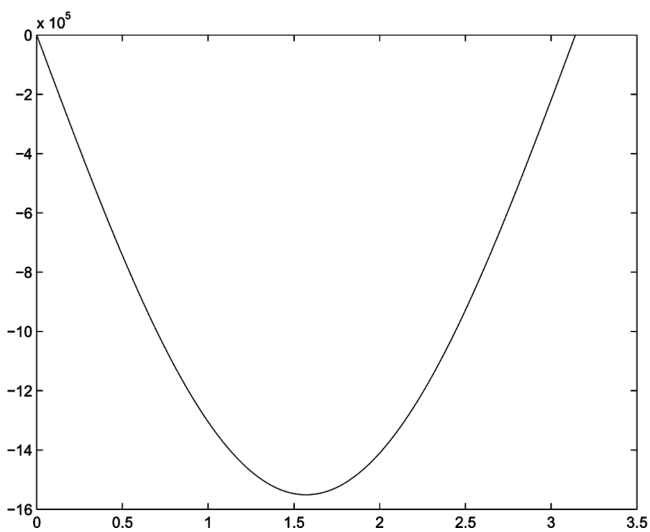

FIGURE 2 $U_h(x)$ $(n = 1000, h = \pi/1001)$.

Similar results are also obtained for FEM. Existence of such ghost solutions implies that a verification of solution is necessary in numerical computation.

Finally, we remark that, if the problem (1.1) has no solution and if it is solved by FDM and FEM, then ghost solutions as in the above example appear. The proof will be done by the same analysis.

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