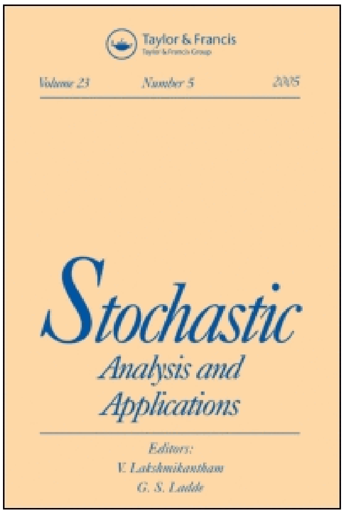

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No Arbitrage and the Growth Optimal Portfolio

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No Arbitrage and the Growth Optimal Portfolio

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Abstract: Recently, several papers have expressed an interest in applying the Growth Optimal Portfolio (GOP) for pricing derivatives. We show that the existence of a GOP is equivalent to the existence of a strictly positive martingale density. Our approach circumvents two assumptions usually set forth in the literature: 1) infinite expected growth rates are permitted and 2) the market does not need to admit an equivalent martingale measure. In particular, our approach shows that models featuring credit constrained arbitrage may still allow a GOP to exist because this type of arbitrage can be removed by a change of numéraire. However, *if* the GOP exists the market admits an equivalent martingale measure under *some* numéraire and hence derivatives can be priced. The structure of martingale densities is used to provide a new characterization of the GOP which emphasizes the relation to other methods of pricing in incomplete markets. The case where GOP denominated asset prices are strict supermartingales is analyzed in the case of pure jump driven uncertainty.

Keywords: Arbitrage; Growth optimal portfolio; Market price of risk; Sigma martingale density.

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1. INTRODUCTION

A Growth Optimal Portfolio (GOP) is a strategy θ^* which maximizes the growth rate of wealth over some time horizon T and relative to some numéraire. Hence, θ^* is usually defined as the strategy which maximizes the expression

$$\mathbb{E}\log\left(W_T^{x_0,\theta}\right),$$

where $W_T^{x_0,\theta}$ denotes the terminal wealth of some trading strategy θ with initial wealth x_0. This definition implicitly assumes that the growth is measured relative to a zero growth rate. To emphasize that a growth rate is a relative term, we define the GOP as an admissible portfolio θ^* satisfying

$$\mathbb{E}\log\left(\frac{W_T^{x_0,\theta}}{W_T^{x_0,\theta^*}}\right) \leq 0$$

for any other admissible portfolio θ. This GOP definition is numéraire independent and this property will be important for our approach. If a numéraire with a negative growth rate is chosen, the expected growth rate of the other assets may explode in comparison: A defaultable bond with a high "loss given default" can lead to this phenomenon.

We consider the following questions:

1. Under what conditions does the GOP exist?
2. How can the GOP be computed?
3. When is the GOP related to a *martingale density*?

Answers to these questions form the main results of this paper:

1. There is a GOP if and only if there is a strictly positive martingale density. This density does not have to define an equivalent martingale measure and the expected growth rate of the GOP does not have to be finite with respect to the given numéraire. We give an example from the literature illustrating this feature.
2. The GOP can be described by a set of nonlinear equations which are closely related to the structure of local martingales existing on the given probability space. Simple solutions of these equations exist only if asset prices are continuous or if the market is sufficiently close to completeness.
3. In an incomplete market the GOP may not be the inverse of a martingale density if the market price of jump risk is high when compared to the risk of facing a large negative drop in the asset price.

The problem of constructing a GOP is classical and has its roots in the theory of gambling. In this sense it dates back to [1] who proves that the GOP outperforms any other strategy in an almost sure sense. This property makes the GOP applicable for asset allocation decisions and a large body of the GOP literature studies this aspect. Long [2]. investigates the GOP's relationship with the numéraire portfolio, a portfolio which has the property that all other asset prices measured in terms of this portfolio are martingales and [2] concludes that the GOP and the numéraire portfolio are identical. The numéraire property of the GOP is further elaborated in for example [3–6]. It should be noted that the relation between the GOP and the numéraire portfolio suggested by [2] does not hold in full generality. This is due to the fact that GOP denominated prices may be strict supermartingales and in this case a numéraire portfolio does not exist. One may generalize the numéraire portfolio concept slightly to establish equivalence, as done in [5] who calls numéraire portfolio any portfolio so that assets measured relative to such portfolio becomes supermartingales. A different generalization was done by [7] where in a discrete time framework generalized numéraire portfolios as in [2] are introduced, but without the usual requirement that they must be self-financing.

The numéraire property of the GOP allows it to be used for the purpose of pricing derivative securities. Platen [8] argues that this can be done even in models where an equivalent martingale measure (EMM) is absent and has developed a benchmark framework to do so, see e.g., [9] and [10]. Essentially this is because the existence of an equivalent martingale measure depends on the selected numéraire. The insight that a given model does not admit an EMM, at least when the risk free asset is used as numérarie, but still allows arbitrage free pricing and hedging is reasonably new. It is noted in [11] in the context of diversity and [12] show that absence of an EMM can be consistent with economic equilibrium. But how can we find a numéraire that provides an EMM, if such is not given from the outset? Our contribution is to show that pricing derivatives can be done *exactly* when the GOP exists. Although existence of a GOP is not equivalent to the existence of a classical risk neutral measure, the GOP existence does imply the existence of at least one numéraire under which an EMM is admitted. The main obstacle is that GOP denominated prices can be strict supermartingales as exemplified by [13] and consequently the original measure does not need to be a martingale measure when the GOP is used as the numéraire. The fact that the GOP deflated market allows an EMM closely relates our result to the First Fundamental Theorem of Asset Pricing (FFTAP) originating with the articles [14] and [15]. FFTAP states that no arbitrage, suitably defined, implies the existence of an equivalent martingale measure (EMM). This equivalence was completely settled for price processes modeled by semimartingales in the articles [16] and [17]. Their proof uses the notion of No Free Lunch with Vanishing Risk (NFLVR) in order to guarantee the existence of a pricing measure. NFLVR is equivalent to having 1) no arbitrage in the sense that no lower bounded strategy with zero initial cost can provide a strictly positive payoff and 2) existence of a strictly positive martingale density, see [18]. We call such an arbitrage a *credit constrained arbitrage*. Delbaen and Schachermayer [18] illustrates that this arbitrage concept is numéraire dependent and consequently FFTAP depends on the choice of the numéraire. In this respect, our result can be seen as a numéraire invariant generalization of the FFTAP. Whereas the FFTAP takes the numéraire as given and ask whether there is an EMM, we ask whether one may find a numéraire under which there is an EMM. The derivation of our main theorem shows that the no credit constrained arbitrage restriction in the FFTAP is redundant in the context of asset pricing. It is merely a feature of the chosen numéraire.

In the previous literature, e.g., [5] and [19], the proof of GOP existence is based on the duality result of [13]. Their result guarantees the existence of a GOP but assumes 1) the market allows an EMM and 2) finite expected growth rates. Our proof applies their result to a market with a carefully chosen numéraire and utilizes an approximation procedure to construct the GOP when expected growth rates are infinite. Independent of our paper, which is based on [20], the working paper [21] has obtained a similar existence result, although their analysis is different as their approach is based on the semimartingale characteristic triplet.

In order to characterize the GOP, we employ a result in [22] characterizing all martingale densities. This characterization uses a process known as the market price of risk. The market price of risk translates uncertainty into expected returns and this process is shown to exist under the condition that there is no arbitrage among strictly positive portfolios. This arbitrage assumption is weaker than demanding no credit constrained arbitrage and is numéraire independent. The characterization of densities implies a simple GOP characterization. In a general setting it is difficult to compute the growth optimal strategy: [19] and [23] provide a general set of Hamilton–Jacobi–Bellman equations. Our approach is to be viewed as an alternative which exploits the structure of martingale densities and links the solution to the fundamental variable in the economy given by the market price of risk. We apply this characterization in a pure jump setting and provide a relatively easy-to-check criterion of when the inverse of the GOP is a strictly positive martingale density. Moreover, we show that the GOP fails to be an inverse martingale density in incomplete markets with a high price of jump risk for jumps near zero.

2. ASSETS AND TRADING STRATEGIES

In this section the basic underlying mathematical framework is presented. We represent uncertainty by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions of right-continuity and augmentation by $\mathbb{P}$-null sets of $\mathcal{F}$. We let $\mathcal{F}_0$ contain the $\mathbb{P}$-null subsets of $\mathcal{F}$ and their complements. $\mathbb{E}$ denotes the expectation operator under the "objective" probability measure $\mathbb{P}$. The time horizon is assumed to be given by some finite $T > 0$. The space of $\mathcal{F}_T$ measurable random variables with the topology induced by convergence in probability is denoted L^0 and the cone of nonnegative $\mathcal{F}_T$ measurable random variables is denoted by L^0_+.

The market is represented by a $d+1$ dimensional semimartingale $S_t = \{(S^0_t, \dots, S^d_t), t \in [0, T]\}$. We follow [24] and denote the set of S-integrable random variables by $L(S)$.

Definition 2.1. A strategy is a predictable process, $\theta_t = \{(\theta^0_t, \dots, \theta^d_t), t \in [0, T]\}$, where $\theta \in L(S)$. Such a strategy is called admissible if the corresponding wealth process satisfies the self-financing condition

$$W^{x_0,\theta}_t \triangleq \sum_{i=0}^{d} \theta^i_t S^i_t = x_0 + \int_0^t \theta_s \, dS_s = x_0 + (\theta \cdot S)_t$$

and is nonnegative for all $t \in [0, T]$. The set of nonnegative portfolio strategies is denoted by $\underline{\Theta}(S)$ and the subset of all strictly positive portfolio strategies is denoted by $\underline{\Theta}^+(S)$. For $\theta \in \underline{\Theta}^+(S)$ the corresponding portfolio fractions are defined by the predictable processes

$$\pi^{\theta^i}_t \triangleq \frac{\theta^i_t S^i_{t-}}{W^{x_0,\theta}_{t-}}, \quad \text{for } i \in \{0, 1, \dots, d\}.$$

We also define

$$\mathcal{K}(x_0) \triangleq \{W^{x_0,\theta}_T \mid \theta \in \underline{\Theta}(S)\}$$

as the space of possible outcomes at time, T, with x_0 as initial wealth.

When no confusion is likely to arise, the notation π^i will be used instead of π^{θ^i}. Given $\theta^1, \dots, \theta^d$ we can specify θ^0 such that the wealth process corresponding to $(\theta^0, \dots, \theta^d)$ is self-financing. Note that the definition of admissibility is invariant under a change of numéraire since the self-financing property is numéraire invariant (see [23], Proposition 2.1).

Several definitions of the growth optimal property are available in the literature. Our definition is adapted from [25] and it allows the GOP to exist even if the *log*-investor can trade to infinite utility. We exemplify that this phenomenon indeed can occur in a simple model.

Definition 2.2. A strategy $\theta^* \in \underline{\Theta}^+(S)$ is called growth optimal, if for any $\theta \in \underline{\Theta}(S)$ it holds that:

$$\mathbb{E}\log\left(\frac{W_T^{x_0,\theta}}{W_T^{x_0,\theta^*}}\right) \leq 0.$$

Remark 2.3. A modification of [5], Remark 2 shows that the mean value in Definition 2.2 is well-defined for any portfolio $\theta \in \underline{\Theta}(S)$. The mean value is well-defined if $\frac{W_T^{x_0,\theta}}{W_T^{x_0,\theta^*}}$ is bounded away from zero. For $\epsilon \in (0,1)$ we define $W_{t,\epsilon}^{x_0,\theta}$ to be:

$$W_{t,\epsilon}^{x_0,\theta} \triangleq \epsilon W_t^{x_0,\theta^*} + (1-\epsilon) W_t^{x_0,\theta}.$$

By definition of θ^* we see that $\mathbb{E}\log \frac{W_{T,\epsilon}^{x_0,\theta}}{W_T^{x_0,\theta^*}} \in [\log(\epsilon), 0]$. The concavity of log yields:

$$\left(\log \frac{W_{T,\epsilon}^{x_0,\theta}}{W_T^{x_0,\theta^*}}\right)^+ \geq (1-\epsilon)\left(\log \frac{W_T^{x_0,\theta}}{W_T^{x_0,\theta^*}}\right)^+,$$

which in turn implies that

$$\mathbb{E}\left(\log \frac{W_T^{x_0,\theta}}{W_T^{x_0,\theta^*}}\right)^+ < \infty.$$

Therefore, the expectation in Definition 2.2 is well-defined with values in $[-\infty, 0]$. In particular, choosing $\theta \triangleq (x_0, 0, \ldots, 0)$ implies that $\mathbb{E}\log\left(W_T^{x_0,\theta^*}\right)$ is well-defined with values in $[0, \infty]$. Hence, Definition 2.2 covers the case where the expected growth rate is infinite.

Clearly, some sort of restrictions on the model must be present for the GOP to exist.

Definition 2.4. An arbitrage is an admissible portfolio constructed from zero initial wealth having a strictly positive value with strictly positive probability at time, T. Hence, no arbitrage is the condition

$$\mathscr{K}(0) \cap L_+^0 = \{0\}.$$

In contrast, we call a self-financing strategy, such that $W_t^{0,\theta} \geq -K$, $\mathbb{P}(W_T^{0,\theta} \geq 0) = 1$ and $\mathbb{P}(W_T^{0,\theta} > 0) > 0$ for some $K > 0$, a credit constrained arbitrage.

When we speak of arbitrage henceforth, we mean arbitrage as defined here. We note that this definition of no arbitrage is weaker than the definition of no arbitrage usually set forth in the literature, see e.g., [16]. This usual type of arbitrage is what we call credit constrained arbitrage since it assumes the investor to have access to some sort of credit. We use the term "credit constrained" to emphazise the difference. Let us comment on why we only want to disregard arbitrages and not credit constrained arbitrages. Firstly, the definition of credit constrained arbitrage is numéraire dependent and in most models it is possible to choose a numéraire which implies arbitrage in this sense, see [18]. In particular, arbitrage can be introduced by applying the risk free asset if this asset is a "poor investment." Secondly, the assumption of no credit constrained arbitrage precludes models in which a GOP does exist. In fact, since the GOP is numéraire independent, when defined carefully, it is clear that the absence of a credit constrained arbitrage is unrelated to existence of the GOP. In contrast, the arbitrage definition given in Definition 2.4 is a necessary requirement to form a GOP.

Assumption 2.5. There is no arbitrage in the financial market.

To have a nontrivial problem we specify an assumption which requires at least one GOP candidate to exist:

Assumption 2.6. The set of admissible strategies is nonempty and there is at least one admissible strategy, θ, such that $W_T^{1,\theta} > 0$ almost surely.

Assumptions 2.5 and 2.6 will be assumed throughout the article.

Remark 2.7. The wealth process corresponding to the portfolio in Assumption 2.6 is strictly positive: if for some $t < T$ we have $\mathbb{P}(W_t^{x_0,\theta} = 0) > 0$, arbitrage is present. Hence, $\underline{\Theta}^+(S) \neq \emptyset$.

Any strictly positive wealth process is called a numéraire. Since the GOP definition we will use is numéraire independent, we can change the numéraire in the market without affecting our results. Consequently, without loss of generality, we can take the first asset as the numéraire:

$$S_t^0 \equiv S_0^0 \triangleq 1.$$

Our approach does not require S^0 to be the locally risk free asset, hence the existence of a such asset is not assumed. Perhaps surprisingly, we can assume without loss of generality that S^i is bounded. To see this, replace S^0 by the wealth process $W^{x_0,\theta'}$ where θ' holds an equal number of each asset i. We can use this portfolio as the numéraire without changing the set of admissible wealth processes. However, in this equivalent deflated market all price processes of the remaining securities are bounded. Therefore, S^i can be taken as a special semimartingale and we use the canonical representation $S_t^i = S_0^i + \overline{A}_t^i + \overline{M}_t^i$, where $\overline{A}^i$ is a predictable process of finite variation. Given the boundedness of S^i, we see from [26], Lemma I.4.24, that $\overline{M}^i$ is also local bounded and, therefore, is a locally square integrable local martingale. This implies that $[\overline{M}^i]$ is locally integrable and that the quadratic characteristic/angle brackets $\langle \overline{M}^i \rangle$ exist as predictable processes. The process $\langle \overline{M}^i \rangle$ is the compensator of $[\overline{M}^i]$ meaning that $[\overline{M}^i] - \langle \overline{M}^i \rangle$ is a local martingale.

3. DENSITIES

Given the possible discontinuities in the asset prices, we know that integrals with respect to the local martingale part $\overline{M}$ are not local martingales themselves but only sigma martingales. Sufficient conditions for such processes to be local martingales or even genuine martingales are given in [24], see also [17] and [19]. As in [17] it is therefore natural to introduce sigma martingale densities:

Definition 3.1. Let Z be an adapted process such that for $i \in \{0, \ldots, d\}$ the process

$$\widetilde{S}_t^i \triangleq S_t^i Z_t \tag{1}$$

is a sigma martingale. If Z is nonnegative Z is called a *sigma martingale density*, otherwise Z is called a *signed sigma martingale density*. If Z is strictly positive we call Z a *strict sigma martingale density*. The set of sigma martingale densities is denoted by $\mathcal{M}$ and the subset of strict sigma martingale densities is denoted by $\mathcal{M}_+$.

The existence of a sigma martingale density is numéraire independent: if Z is a sigma martingale density for S, then $ZW^{1,\theta}$ is a sigma martingale density for $\frac{S}{W^{1,\theta}}$. We note that a strict sigma density is the density of an EMM if and only if $\mathbb{E}[Z_t] = 1$ for all t.

The admissibility concept we apply demands non-negativity of the wealth process which is in contrast to boundedness from below. This distinction leads to the following lemma:

Lemma 3.2. *For $\theta \in \underline{\Theta}(S)$ the process*

$$\widetilde{W}_t^{x_0,\theta} \triangleq W_t^{x_0,\theta} Z_t \tag{2}$$

is a nonnegative local martingale and, hence, a supermartingale for any sigma martingale density $Z \in \mathcal{M}$.

Proof. From [23], Proposition 2.1 we have $\theta^i \in L(\widetilde{S}^i)$ for $i \in \{0, \ldots, d\}$ and

$$d\widetilde{W}_t^{x_0,\theta} = \sum_{i=0}^{d} \theta_t^i \, d\widetilde{S}_t^i.$$

Since the class of sigma martingales is linear and closed under integration, see e.g., [19], Lemma 7.10, it follows that $\widetilde{W}^{x_0,\theta}$ is a sigma martingale. Hence, from [27] and the fact that wealth processes are nonnegative, the wealth process, $\widetilde{W}_t^{x_0,\theta}$, must itself be a local martingale and by Fatou's Lemma it is also a supermartingale. □

One might as well refer to sigma martingale densities as supermartingale densities when admissibility is restricted to nonnegative wealth processes. Becherer [5] notes that the set of sigma martingale densities is a subset of the set of supermartingale densities and as we shall see the difference between the two concepts is important. The concept of a supermartingale density, defined by substituting sigma martingale with supermartingale in Definition 3.1, is only in special cases relevant for asset pricing purposes since this family includes a wide spectrum of processes with no connection to the traditional martingale densities. If the investor faces a portfolio restriction, it is well-known that the investor's marginal utility in optimum can correspond to a supermartingale density. The presence of jumps in asset prices endogenously forces such constraints on the investor's portfolio choice. In Section 8 we will discuss this aspect.

4. PROPERTIES OF THE GOP

The following theorem shows that the "usual" growth optimal properties carry over to the generalizing GOP of Definition 2.2. These will be needed for our main result.

Theorem 4.1. *If $\theta^* \in \underline{\Theta}^+(S)$ is a GOP strategy, then*:

1. *The corresponding wealth process $W_t^{x_0,\theta^*}$ has the numéraire property*:

$$\frac{W^{x_0,\theta}}{W^{x_0,\theta^*}} \quad \text{is a supermartingale for all } \theta \in \underline{\Theta}(S).$$

2. *If $\tilde{\theta}^* \in \underline{\Theta}^+(S)$ is another GOP, then $W^{x_0,\tilde{\theta}^*} = W^{x_0,\theta^*}$.*
3. *If $\theta \in \underline{\Theta}^+(S)$ has the numéraire property, then θ is growth optimal.*

Proof. If the expected growth rate of the GOP is finite, then by choosing $\theta \triangleq (x_0, 0, \ldots, 0)$ it is evident that θ^* solves the *log*-investor's problem and hence θ^* has the numéraire property, see e.g., Becherer [5]. If this is not the case, consider a market that uses $W_t^{x_0,\theta^*}$ as numéraire. In this market all growth rates are bounded (by zero) and we can apply the previous argument here. Since the numéraire property is invariant to shifts of numéraire, and changing the numéraire of both the denominator and the numerator leaves the fraction unchanged, the first part follows. If the strategies θ^* and $\tilde{\theta}^*$ are both growth optimal, then by the preceding part of the theorem, they must both have the numéraire property. In this case,

$$\frac{W^{x_0,\tilde{\theta}^*}}{W^{x_0,\theta^*}} \quad \text{and} \quad \frac{W^{x_0,\theta^*}}{W^{x_0,\tilde{\theta}^*}}$$

are both supermartingales. Jensen's inequality yields that both processes must be constant and the second part follows.

The last claim follows by the same change of numéraire, which allows this case to be handled as if growth rates were finite. Hence, the proof follows from standard arguments. □

Corollary 4.2. *θ^* is the GOP strategy, if and only if for any admissible outcome $W^{x_0,\theta}$ we have*

$$\mathbb{E}\left[\frac{W_T^{x_0,\theta}}{W_T^{x_0,\theta^*}}\right] \leq 1.$$

The following simple example illustrates a model with infinite expected growth rates but the GOP defined by Definition 2.2 exists:

Example 4.3. Let M be a continuous local martingale such that $\sup_{0\leq t\leq T} \mathbb{E}[M_t^2] < \infty$ and consequently both $\mathbb{E}[M_T]$ and $\langle M\rangle_T$ are well-defined but the last process does not need to be integrable. We assume that $\mathbb{E}[\langle M\rangle_T] = \infty$ and define $A_t \triangleq \langle M\rangle_t$, $S_t \triangleq (1, S_t^1)$ and $S_t^1 = \mathscr{E}(A+M)_t$. Then

$$\mathbb{E}\log\left(S_T^1\right) = \mathbb{E}\log(\mathscr{E}(M+A)_T) = \mathbb{E}[M_T + 0.5\langle M\rangle_T] = \infty. \tag{3}$$

Therefore, the problem of maximizing *log*-utility has no solution in the usual sense. However, the GOP exists according to the generalized definition; simply place all the wealth in the stock. This can be verified by applying Itô's formula to see that $\frac{W^{x_0,\theta}}{S^1}$ is a local martingale for all $\theta \in \underline{\Theta}(S)$. Consequently S^1 has the numéraire property and the claim follows from last part of Theorem 4.1.

An example of such a continuous local martingale can be constructed from the 3 dimensional Bessel process:

$$dX_t \triangleq dB_t + \frac{1}{X_t}dt, \quad X_0 \triangleq 1.$$

The inverse of this process, $M_t \triangleq \frac{1}{X_t}$, constitutes an example of a genuine local martingale satisfying $\mathbb{E}[M_t^2] < \infty$ but $\mathbb{E}[\langle M \rangle_t] = \infty$ for all t (see e.g., [28], Proposition IV.1.23 and Exercise V.2.13).

5. THE MAIN RESULT

This section is devoted to prove our central result:

Theorem 5.1. *The following conditions are equivalent*:

1. *The GOP exists in the sense of Definition* 2.2.
2. *There exists a strictly positive sigma martingale density,* $\mathcal{M}_+ \neq \emptyset$.
3. *There exists a numéraire under which there is an EMM.*

This explains why it is possible to price assets even when the market does not allow a risk neutral measure and arbitrage opportunities appear to exist. This situation has been encountered in the benchmark framework, see [29] and [8], and the phenomenon is also present in the articles on diversity, see [30] and [11]. In both cases, credit constrained arbitrage opportunities exists, when using the risk-free asset as a numérarire, because a strategy exists which dominates the risk-free asset in the long run, i.e. the risk-free asset is not a maximal element as defined below. Nevertheless, the GOP exists and by using the GOP as the numéraire there are no longer any credit constrained arbitrage opportunities. Hence, we conclude that such models are indeed theoretically consistent with arbitrage free pricing and the fact that the risk-free asset is not maximal may be interpreted to reflect the folklore that "stocks are better than bonds" in the long run.

The rest of this section is devoted to proving Theorem 5.1 and we start with a few auxiliary observations. A maximal element for the marketed subspace $\mathcal{K}(1)$ is an outcome $W_T^{1,\theta'}$ of an admissible portfolio θ' such that the following implication holds for $\theta \in \mathcal{K}(1)$:

$$W_T^{1,\theta'} \leq W_T^{1,\theta} \Rightarrow W_T^{1,\theta} = W_T^{1,\theta'} \quad a.s.$$

Consequently, maximal elements are outcomes of those strategies which cannot be outperformed systematically.

Remark 5.2. Note that [18] define a maximal element as an outcome, $(H \cdot S)_T$, of some strategy H, which cannot be dominated and where $(H \cdot S)_t$ is lower bounded. This is equivalent to our definition, which merely requires the investor to have sufficient initial wealth for the strategy H to be admissible.

An *approximate arbitrage* is a sequence of admissible strategies $(W^{\frac{1}{n},\theta_n})_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} W_T^{\frac{1}{n},\theta_n} = f$$

exists in probability for some random variable $f : \Omega \to [0, \infty]$, which is strictly positive with strictly positive probability.

Lemma 5.3.

1. *The terminal GOP value is a maximal element in $\mathcal{K}(1)$.*
2. *If a GOP exists, then there is no credit constrained arbitrage in the GOP denominated market.*
3. *If there is an approximate arbitrage, a GOP does not exist.*

Proof. (1) This is evident from the definition of the GOP. (2) From Remark 2.7 we have that W_t^{1,θ^*} is strictly positive for all $t \in [0, T]$ and can be used as a numéraire. Let θ be a credit constrained arbitrage in this market:

$$\frac{W_t^{0,\theta}}{W_t^{1,\theta^*}} \geq -1 \quad \forall t \in [0, T].$$

By rearranging we get a contradiction to the maximality of θ^*. (3) Let $(W^{\frac{1}{n},\theta_n})_{n\in\mathbb{N}}$ be an approximate arbitrage and let f be the limit as $n \to \infty$. We define the set

$$A \triangleq \{\omega \in \Omega \mid f(\omega) > 0\},$$

and by assumption $\mathbb{P}(A) > 0$ and so $\mathbb{E}[f]$ is strictly positive and well-defined although possibly diverging to plus infinity. From Fatou's Lemma and the fact that GOP denominated wealth processes are supermartingales we get

$$0 < \mathbb{E}\left[\frac{f}{W_T^{1,\theta^*}}\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\frac{W_T^{\frac{1}{n},\theta_n}}{W_T^{1,\theta^*}}\right] \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

which contradicts the existence of the GOP. □

Proof of Theorem 5.1. (1) ⇒ (2) The complication is that the GOP may not be directly related to a sigma martingale density. Suppose there is no strict martingale density but the GOP exists. We will show that this implies a contradiction by looking at the GOP denominated market. Since the set of strict martingale densities is empty, no EMM can exist and therefore the FFTAP yields the existence of a FLVR.

From Lemma 5.3 there is no credit constrained arbitrage and [16], Proposition 3.6, implies the existence of a sequence of $\frac{1}{n}$-admissible portfolios θ_n, such that $W_T^{\frac{1}{n},\theta_n} \to f$ with $\mathbb{P}(f \geq 0) = 1$ and $\mathbb{P}(f > 0) > 0$. This is an approximate arbitrage and consequently the GOP cannot exist by the last property listed in Lemma 5.3.

(2) ⇒ (3) We first define the set

$$\mathscr{C}(1) \triangleq \left\{ f \in L^\infty(\mathbb{P}) : f \leq W_T^{1,\theta} \text{ for some } \theta \in \underline{\Theta}(S) \right\}.$$

The following is a central feature of $\mathscr{C}(1)$ when a density is present.

Lemma 5.4. *If there is a strictly positive sigma martingale density, $\mathscr{M}_+ \neq \emptyset$, then $\mathscr{C}(1)$ is bounded in probability and closed in the topology induced by convergence in probability.*

Proof. That the set $\mathscr{C}(1)$ is bounded in probability is part of the proof of [18], Theorem 4. That $\mathscr{C}(1)$ is closed in probability is the essential content of [31], Theorem D: defining $\widetilde{\mathscr{C}}(1) \triangleq \{Y : Y = X_T Z_T \;\text{ for }\; X \in \mathscr{C}(1)\}$ where $Z \in \mathscr{M}_+$. It follows that $\widetilde{\mathscr{C}}(1)$ is a closed subset of L^0 and so $\mathscr{C}(1)$ is also closed in probability since the density Z is strictly positive. □

Lemma 5.4 and [16], Lemma 4.3 imply the existence of a maximal element belonging to $\mathscr{K}(1)$. A change of numéraire combined with [18], Theorems 4 and 10 yields the desired implication.

(3) ⇒ (1) Without loss of generality we can assume that $\mathbb{P}$ is an EMM for the market. The problem remaining is that expected growth rates can be infinite. We, therefore, construct a sequence of approximative solutions and show that such a sequence converges to the GOP. The approximation procedure utilizes the functions

$$f_n(x) \triangleq \ln(x) 1_{(x \leq n)} + g_n(x) 1_{(x > n)}$$

where g_n is bounded such that f_n is two times continuous differentiable satisfying the Inada conditions and g_n' is a convex function less than $\frac{1}{x}$. Trivially $f_n(x) \to \ln(x)$ as $n \to \infty$ for all $x > 0$. The following is a consequence of the main result in [13]:

Theorem 5.5. *For all $n \in \mathbb{N}$, there is an admissible outcome $X_T^n \triangleq W_T^{1,\theta^n}$ such that*

$$X_T^n \in \arg \sup_{\theta \in \underline{\Theta}(S)} \mathbb{E}\left[f_n\left(W_T^{1,\theta} \right) \right]$$

and $X_T^n = I_n(y_n Y_T^n) > 0$ for some supermartingale density Y^n, $Y_0^n = 1$, where I_n is the inverse of f_n' and y_n is a Lagrange multiplier.

Note that $I_n(x)$ is simply x^{-1} for $x \le n$. We will use the observation that the Lagrange multipliers satisfy $y_n \le 1$. This follows by the optimality of X^n and since $f_n'(x) \le \frac{1}{x}$

$$y_n = \mathbb{E}\big[y_n Y_T^n X_T^n\big] = \mathbb{E}\big[f_n'(X_T^n)X_T^n\big] \le 1.$$

The sequence $\{X^n\}_{n\in\mathbb{N}}$ obtained in Theorem 5.5 may not converge but *if* it does, the limit is a natural GOP candidate. However, by combining [16], Lemma A.1.1 and Lemma 5.4 we get a sequence $\{\widetilde{X}^n\}_{n\in\mathbb{N}}$ with $\widetilde{X}^n \in \operatorname{conv}\{X^n, X^{n+1}, \dots,\}$ such that $\widetilde{X}^n$ converges in probability to a finite valued random variable, X^*, where X^* is the outcome of some admissible strategy. Since any sequence converging in probability contains a pointwise convergent subsequence, we may and do assume that $\widetilde{X}^n$ converges to X^* almost surely.

The proof is concluded by showing that the portfolio generating the terminal outcome X^* is the GOP. By Corollary 4.2 it suffices to show that for any $X \triangleq W_T^{x_0,\theta}$ with $\theta \in \underline{\Theta}(S)$ we have

$$\mathbb{E}\left[\frac{X}{X^*}\right] \le 1.$$

We assume first that X is bounded. From Fatou's Lemma, and since $f_n'(x) \to x^{-1}$ for all $x > 0$ we have

$$\mathbb{E}\left[\frac{X}{X^*}\right] = \mathbb{E}\big[X \lim_{n\to\infty} f_n'(\widetilde{X}^n)\big] \le \liminf_{n\to\infty} \mathbb{E}\big[Xf_n'(\widetilde{X}^n)\big].$$

Hence, we need to consider the last limit. Since $\widetilde{X}^n$ belongs to the $\operatorname{conv}\{X^n, X^{n+1}, \dots,\}$ there are nonnegative constants $\{\lambda_k\}_{k=n}^{\infty}$ adding to one such that $\widetilde{X}^n = \sum_{k=n}^{\infty} \lambda_k X^k$. We get the following chain of inequalities:

$$\begin{aligned}
\mathbb{E}\big[Xf_n'(\widetilde{X}^n)\big] &= \mathbb{E}\left[Xf_n'\left(\sum_{k=n}^{\infty}\lambda_k X^k\right)\right] \\
&\le \mathbb{E}\left[X\sum_{k=n}^{\infty}\lambda_k f_n'(X^k)\right] \\
&= \mathbb{E}\left[X\sum_{k=n}^{\infty}\lambda_k f_k'(X^k)\right] + \mathbb{E}\left[X\sum_{k=n}^{\infty}\lambda_k\big(f_n'(X^k) - f_k'(X^k)\big)\right] \\
&= \sum_{k=n}^{\infty}\lambda_k \mathbb{E}\big[Xf_k'(X^k)\big] + \mathbb{E}\left[X\sum_{k=n}^{\infty}\lambda_k\big(f_n'(X^k) - f_k'(X^k)\big)\right] \\
&\le \sum_{k=n}^{\infty} y_k\lambda_k + \mathbb{E}\left[X\sum_{k=n}^{\infty}\lambda_k\big(f_n'(X^k) - f_k'(X^k)\big)1_{\{X^k\ge n\}}\right] \\
&\le 1 + \mathbb{E}\left[X\sum_{k=n}^{\infty}\lambda_k\frac{1}{X^k}1_{\{X^k\ge n\}}\right] \\
&\le 1 + \frac{\mathbb{E}[X]}{n} \to 1.
\end{aligned}$$

Since the elements of the sum on the right-hand side of the fourth inequality are all zero whenever $X^k \leq n,\ k \in \{n, n+1, \ldots, \}$. So if X is bounded and dominated by an admissible outcome it holds that $\mathbb{E}[\frac{X}{X^*}] \leq\ 1$. For a general outcome X, we can apply the above with $X_M \triangleq X \wedge M$. The general case then follows from an application of the Monotone Convergence Theorem with $M \to \infty$. □

6. THE MARKET PRICE OF RISK

This section provides a description of sigma martingale densities in the present setup. These densities are described by a market price of risk process and we connect the time at which a martingale density reaches zero with an explosion time for the GOP. The market price of risk process provides a relation between the martingale part and the drifts of the risky securities. Basically, this process determines the expected return for holding some amount of risk where risk is quantified by the martingale M's quadratic characteristic $\langle M \rangle$. Under different assumptions the market price of risk has been analyzed in the literature (see e.g., [22, 32, 33]). For simplicity, we consider the case of only two assets, one of which is used as numéraire: $S = (S^0, S^1) = (1, 1 + \overline{A} + \overline{M})$ with S_t^1 strictly positive and bounded for $0 \leq t \leq T$.

Delbaen and Schachermayer [33] show under the assumption of no credit constrained arbitrage the existence of a predictable process λ such that

$$\frac{d\overline{A}_t}{d\langle \overline{M} \rangle_t} = \lambda_t. \tag{4}$$

This relationship is still true under the assumption of no arbitrage in the sense of Definition 2.4. To see this simply note that the arbitrage portfolio constructed in [33], Theorem 3.5 is also a nonnegative process and, hence, an arbitrage in the strong sense. This is formulated in the following lemma:

Lemma 6.1. *There exists a predictable process λ such that*

$$S_t^1 = \overline{A}_t + \overline{M}_t = \int_0^t \lambda_s d\langle \overline{M} \rangle_s + \overline{M}_t.$$

It is easier to work with stochastic exponentials and since S^1 is a strictly positive process we can write $S_t^1 = \mathscr{E}(M + A)_t$ where

$$dA_t \triangleq \frac{1}{S_{t-}^1} d\overline{A}_t \quad \text{and} \quad dM_t \triangleq \frac{1}{S_{t-}^1} d\overline{M}_t.$$

We define the market price of risk process $\hat{\lambda}$ by

$$\hat{\lambda}_t \triangleq \frac{dA_t}{d\langle M\rangle_t} = \lambda_t S^1_{\{t-1\}}.$$

The market price of risk $\hat{\lambda}$ corresponds to pricing in a relative sense, i.e. uncertainty about future returns is translated into a higher expected return, whereas λ is based on the quadratic variation of the stock price itself and hence a measure in absolute terms. Although the absence of arbitrage provides the process $\hat{\lambda}$, we do not know a priori if $\hat{\lambda}$ is M-integrable. This assumption is needed and will be made throughout the remaining sections:

Assumption 6.2. The predictable process $\hat{\lambda}$ is in $L(M)$.

The market price of risk process provides the structure needed to give a precise description of sigma martingale densities. Two local martingales L and M are said to be orthogonal if $[L, M]$ is a local martingale. The following is similar to a result in [34].

Lemma 6.3. *For any martingale L, orthogonal to M, the process given by*

$$Z_t \triangleq \mathscr{E}(-\hat{\lambda}\cdot M + L)_t$$

is a signed sigma martingale density.

Proof. Note that in the general case Z can jump to negativity. By expressing the stochastic exponential Z as a stochastic differential equation we see that

$$dZ_t = -\hat{\lambda}_t Z_{t-} dM_t + Z_{t-} dL_t$$

so $\widetilde{S}^0$ is a sigma martingale. If it is nonnegative it is in fact a supermartingale by the arguments used in Lemma 3.2. The product rule, Theorem 6.1 and [24], Theorem II.29 yield:

$$\begin{aligned} d(S^1_t Z_t) &= S^1_{t-} dZ_t + Z_{t-} dS^1_t + d[Z, S^1]_t \\ &= S^1_{t-} dZ_t + Z_{t-} S^1_{t-} dM_t + Z_{t-} S^1_{t-} dA_t - \hat{\lambda}_t Z_{t-} S^1_{t-} d[M]_t + Z_{t-} S^1_{t-} d[M, L] \\ &= S^1_{t-} dZ_t + Z_{t-} S^1_{t-} dM_t - \hat{\lambda}_t Z_{t-} S^1_{t-} d([M]_t - \langle M\rangle_t) + Z_{t-} S^1_{t-} d[M, L]. \end{aligned}$$

This shows that $\widetilde{S}^1$ is also a sigma martingale. □

Remark 6.4. The market price of risk is numéraire dependent. However, if Z is a sigma martingale density under the numéraire X and Y is another numéraire, then $\widetilde{Z} \triangleq \frac{Y}{X}Z$ is a density under the numéraire Y. So by establishing the structure of such a density under any given numéraire, we infer the density structure under any other numéraire.

To see how the assumption of M-integrability of the market of risk is related to the existence of a GOP, we consider the following example from [33]:

Example 6.5. Assume that the local martingale M is continuous. Absence of arbitrage implies the existence of a unique market price of risk process $\hat{\lambda}$. In this case the GOP strategy is given by $\pi_t^{1^*} \triangleq \hat{\lambda}_t$, see Corollary 7.4 below. Girsanov's Theorem suggests the density of an EMM $\mathbb{Q}$ given by $Z_t \triangleq \mathcal{E}(-\hat{\lambda} \cdot M)_t$. However, the stopping time defined by

$$\tau \triangleq \inf_{t>0}\left\{ \int_0^t \hat{\lambda}_u^2 d\langle M\rangle_u = \infty \right\} \tag{5}$$

is strictly less than T, when $\hat{\lambda} \notin L(M)$. In this case the candidate $\mathbb{Q}$-measure becomes singular with respect to $\mathbb{P}$ after τ, see [33], because

$$\{Z_\tau = 0\} = \left\{ \int_0^\tau \hat{\lambda}_u^2 d\langle M\rangle_u = \infty \right\}.$$

Since the GOP, W_t^{1,θ^*}, equals Z_t^{-1} on $(t < \tau)$ it follows that $W_t^{1,\theta^*} \to \infty$ as $t \to \tau$ meaning that the GOP explodes. The economic intuition of an exploding GOP is that tomorrow will bring so much wealth that getting a dollar today is infinitely more attractive than getting a dollar tomorrow! This implies that pricing assets after time τ loses all meaning.

This example illustrates that the absence of arbitrage is not sufficient to ensure the existence of a non-explosive solution to the *log*-investor's problem because no arbitrage fails to imply $\hat{\lambda} \in L(M)$. This, however, is not "just an integrability assumption," it has a clear economic interpretation as it implies explosion of what would be an otherwise reasonable trading strategy. If one is willing to accept explosive behavior the discussion above implies the following theorem:

Theorem 6.6. *If S is continuous, the GOP θ^* exists up to the stopping time τ defined by* (5) *and the GOP explodes at time τ:*

$$\lim_{t\to\tau} W_t^{1,\theta^*} = \infty \quad a.s.$$

In the above case where the market price of risk fails to be M-integrable we deduce from Theorem 5.1 that there is no strict sigma martingale density for the market.

7. GOP CHARACTERIZATION

This section provides a new approach to the characterization of the growth optimal strategy. The fundamental idea is to exploit the numéraire property and Lemma 6.3 to obtain a constructive proof that sheds light on the GOP structure. This section assumes that GOP denominated prices are local martingales or, stated in the terminology of [13], that the dual optimizer is a strict local martingale density. Therefore the following GOP characterization is presented under a sufficient set of conditions. However, these conditions provide the basis for the discussion of the case where the inverse GOP is not such a density. This discussion is the content of the next section.

The following assumption simplifies the exposition. It can be relaxed at the costs of more complicated statements.

Assumption 7.1. The martingale M is quasi-left-continuous and hence $\langle M\rangle$ has a continuous version, see [26], Theorem I.4.2.

We use this version of $\langle M\rangle$ and consequently A is also a continuous process.

Lemma 7.2. *If there exists $\pi^* \in L(M)$ and a local martingale L^* orthogonal to M, such that for all $t \in [0, T]$ we have*

$$Z_t^* \triangleq \mathscr{E}(-\hat{\lambda}\cdot M + L^*)_t > 0 \tag{6}$$

$$0 = ((\pi^* - \hat{\lambda})\cdot M)_t - (\pi^*\hat{\lambda}\cdot([M] - \langle M\rangle))_t + L_t^* + (\pi^*\cdot[M, L^*])_t, \tag{7}$$

then π^* is the growth optimal fraction to be invested in S^1.

Proof. From Lemma 6.3 we know that Z^* is a sigma martingale density, and combining this with the last part of Theorem 4.1 it suffices to verify that (θ^* corresponds to the fraction π^*)

$$1 \equiv Z_t^* W_t^{1,\theta^*}. \tag{8}$$

According to [24], Theorem II.38, Lemma 6.1 and Equation (6) the right-hand side of Equation (8) equals

$$\mathscr{E}\big(-\hat{\lambda}\cdot M + L^* + \pi^*\hat{\lambda}\cdot\langle M\rangle + \pi^*\cdot M + [-\hat{\lambda}\cdot M + L^*, \pi^*\hat{\lambda}\cdot\langle M\rangle + \pi^*\cdot M]\big)_t.$$

The continuity assumption of $\langle M\rangle$ yields in turn that Equation (8) is equivalent to:

$$1 \equiv \mathscr{E}\big((\pi^* - \hat{\lambda})\cdot M - \pi^*\hat{\lambda}\cdot([M] - \langle M\rangle) + L^* + \pi^*\cdot[M, L^*]\big)_t,$$

and the lemma follows. □

Before a general solution is provided, some simple cases are considered in which a solution can be obtained directly from Equation (7).

Corollary 7.3. *If there exists a martingale, L^*, such that*

$$(\hat{\lambda}^2 \cdot ([M] - \langle M \rangle))_t = L_t^* + (\hat{\lambda} \cdot [M, L^*])_t,$$

and if (6) *holds, then* $\pi_t^* \triangleq \hat{\lambda}_t$ *is the growth optimal fraction to be invested in* S^1.

In general, if X is an arbitrary martingale it might not be possible to find L' such that

$$[X]_t - \langle X \rangle_t = L_t' + [X, L']_t.$$

However, if such a martingale L' exists it must be orthogonal to X. In the continuous case we have $[M] = \langle M \rangle$ so:

Corollary 7.4. *If M is continuous, then* $\pi_t^* \triangleq \hat{\lambda}_t$ *is the growth optimal fraction to be invested in* S^1.

Hence, in the continuous case there is no orthogonal martingale part L^* to M and one is certain that the inverse GOP is a sigma martingale density. Furthermore this density corresponds to the Minimal Martingale Measure whenever it exists, see [34]. It follows that if one chooses the GOP for asset pricing purposes, all risks which are unspanned by the existing assets have a market price of risk equal to zero, see [5], Example 3. This example shows that passing to the incomplete diffusion market means passing from the unique market price of risk to the unique projection of the set of market prices of risk onto the volatility process. Therefore, using the GOP for asset pricing in an incomplete market implicitly equates the market price of non-traded risk to zero.

When jumps are present, the situation is more delicate. A notable exception occurs whenever the martingale M has the following spanning property:

Corollary 7.5. L^* *in Equation* (7) *can be taken to be zero if and only if* $[M] - \langle M \rangle$ *belongs to the stable subspace generated by M meaning that for some predictable process* $f \in L(M)$:

$$[M]_t - \langle M \rangle_t = (f \cdot M)_t.$$

In this case, if the predictable process

$$\pi_t^* \triangleq \frac{\hat{\lambda}_t}{1 - f_t \hat{\lambda}_t}$$

satisfies Equation (6) *and* $\pi^* \in L(M)$, π^* *is the growth optimal fraction to be invested in* S^1.

This is always true in the continuous case ($f \equiv 0$). When jumps are present an example is given by the case of a complete market with simple Poissonian jumps. Here M is a compensated Poisson process in which case $f_t \equiv 1$. This case is studied by e.g., [35]. In general there is such a property if and only if one can choose $\Delta f = (\Delta M)^2$, i.e., if jump sizes are either predictable or if the square of the jump size can be obtained. The last property is only present when the market is complete. This leads to the conclusion that the GOP-induced density is only related to the Minimal Martingale Measure in the continuous case *or* in cases where jump sizes can be hedged.

Returning to the general case, we elaborate on (7) and write

$$M_t = M_t^c + M_t^d \quad 0 \leq t \leq T$$

where M^c is the continuous part and M^d is the purely discontinuous local martingale part of M. According to [26], Theorem I.4.18 this decomposition exists and is unique. Similarly we write

$$L_t^* = L_t^{*c} + L_t^{*d} \quad 0 \leq t \leq T.$$

Since $[M, L^*]$ is a local martingale of finite variation, it must be a pure jump local martingale according to [26], Lemma 4.14(b). Hence, by matching the continuous local martingale terms it follows that

$$L_t^{*c} = -((\pi^* - \hat{\lambda}) \cdot M^c)_t.$$

So the pure discontinuous local martingale part of (7) equals

$$0 \equiv \big((\pi_s^* - \hat{\lambda}_s) \cdot M^d\big)_t - \big(\pi_s^* \hat{\lambda}_s \cdot ([M] - \langle M \rangle)\big)_t + L_t^{*d} + \big(\pi_s^* \cdot [M, L^*]\big)_t. \quad (9)$$

Purely discontinuous local martingales are completely characterized by their jumps, see [26], Corollary I.4.19, so it suffices to calculate the size of the jumps in Equation (9) and equate them to zero

$$\big(\pi_t^* - \hat{\lambda}_t\big)\Delta M_t - \pi_t^* \hat{\lambda}_t \Delta M_t^2 + \Delta L_t^* + \pi_t^* \Delta M_t \Delta L_t^* \equiv 0,$$

which is equivalent to

$$\Delta L_t^* = \Delta M_t \left(\hat{\lambda}_t - \frac{\pi_t^*}{1 + \pi_t^* \Delta M_t} \right).$$

What remains in order to provide a complete characterization of the martingale L^* and the growth optimal strategy is the orthogonality condition between M and L^*, meaning that $\langle M, L^* \rangle_t \equiv 0$. These properties completely characterize the GOP: we need to find a martingale

L^* and a predictable process $\pi^* \in L(M)$ such that (6) holds and

1. $\langle M, L^* \rangle_t \equiv 0$,
2. $\Delta L^*_t = \Delta M_t \big(\hat{\lambda}_t - \frac{\pi^*_t}{1+\pi^*_t \Delta M_t}\big)$,
3. $L^{*c}_t = ((\hat{\lambda} - \pi^*) \cdot M^c)_t$.

These conditions are equivalently described by the following nonlinear equation:

Theorem 7.6. *If there exists a predictable process π^* satisfying*

$$\left\langle M, \Delta M\left(\hat{\lambda} - \frac{\pi^*}{1+\pi^*\Delta M}\right) + (\hat{\lambda} - \pi^*) \cdot M^c \right\rangle_t \equiv 0$$

and π^* is admissible, then π^* is the growth optimal fraction to be invested in S^1.

8. STRICT SUPERMARTINGALE DENSITIES

Examples showing that GOP-denominated prices are supermartingales are given in [5, 13, 36]. It is well-known that this phenomenon can be present when trading is restricted, see e.g., [39]. We give a precise formulation of when this phenomenon occurs in a model with no trading restriction. Corollary 7.4 shows that in this case, strict supermartingality can only happen when jumps are present. The results we derive are related to the results in [40], who studies exponential Levy processes, see also [37].

We specialize to the case of two assets,

$$S^0_t \triangleq 1, \quad S^1_t \triangleq \mathscr{E}(A_t + M_t),$$

where the martingale M is a pure jump martingale. M is constructed by a random measure, μ, which we assume to be without explosions and let $(\mathscr{F}_t)_{0 \le t \le T}$ be the natural filtration generated by μ. The compensating measure for μ is denoted by ϕ. We assume that ϕ satisfies

$$\phi(dy, ds) = \varphi_s(y)\kappa(dy)ds,$$

for an intensity process φ and a probability measure κ on the markspace denoted by $(E, \mathscr{E})$. Unless otherwise stated, κ is without singularities, i.e. $\kappa(\{y\}) = 0 \forall y \in E$. The martingale M is then given by:

$$M_t \triangleq \int_0^t \int_E b_s(y)(\mu - \phi)(dy, ds)$$

where $b > -1$ is some predictable flow. The market price of risk, $\hat{\lambda}$, is some given M-integrable, predictable process and $A_t = (\hat{\lambda} \cdot \langle M \rangle)_t$. We are looking for a process, π^*, such that the equation in Theorem 7.6 is satisfied. In this pure jump case it means that the portfolio fraction, π^*, needs to satisfy

$$\hat{\lambda}_t \int_E b_t^2(y)\varphi_t(y)\kappa(dy) = \int_E \frac{b_t^2(y)\pi_t^*}{1+\pi_t^* b_t(y)}\varphi_t(y)\kappa(dy) \triangleq \Pi_t(\pi_t^*). \tag{10}$$

It is apparent that $\Pi_t(\cdot)$ is increasing but a priori the left-hand side of (10) can be anything.

Lemma 8.1. *Define the max fraction $\bar{\pi}_t$ (positive) and the min fraction $\bar{\pi}_t$ (negative) as*

$$\underline{\pi}_t \triangleq \inf\{\pi \mid 1+\pi b_t(y) > 0 \ \kappa\text{-almost everywhere}\}$$
$$\underline{\pi}_t \triangleq \sup\{\pi \mid 1+\pi b_t(y) > 0 \ \kappa\text{-almost everywhere}\}.$$

Portfolios consisting of min and max fractions are admissible and any portfolio fraction to be held in S^1, π, outside the interval $[\underline{\pi}_t, \bar{\pi}_t]$ leads to $\mathbb{E}\log(W_T^\pi) = -\infty$ under any numéraire. Hence, all candidate GOP fractions are within $[\underline{\pi}_t, \bar{\pi}_t]$.

Proof. Outside the interval $[\underline{\pi}_t, \bar{\pi}_t]$ there is a positive probability of realizing negative values, whereas the portfolio values are nonnegative inside. □

A similar insight is provided in [38], Proposition 1. In the light of Equation (10) and Lemma 8.1 we define the following predictable subsets of $\Omega \times [0, T]$:

$$\underline{C} \triangleq \left\{(t,\omega) \mid \Pi_t(\underline{\pi}_t) > \hat{\lambda}_t \int_E b_t^2(y)\varphi_t(y)\kappa(dy)\right\}$$
$$C \triangleq \left\{(t,\omega) \mid \Pi_t(\underline{\pi}_t) \le \hat{\lambda}_t \int_E b_t^2(y)\varphi_t(y)\kappa(dy) \le \Pi_t(\overline{\pi_t})\right\}$$
$$\overline{C} \triangleq \left\{(t,\omega) \mid \Pi_t(\overline{\pi_t}) < \hat{\lambda}_t \int_E b_t^2(y)\varphi_t(y)\kappa(dy)\right\},$$

where the functional Π_t is defined by (10). We note that the following asymptotes are possible:

$$\lim_{\pi\to\overline{\pi_t}} \Pi_t(\pi) = \infty, \quad \lim_{\pi\to\underline{\pi_t}} \Pi_t(\pi) = -\infty.$$

Theorem 8.2. *The predictable process, π^*, defined as*

$$\pi^* \triangleq \begin{cases} \underline{\pi} & \text{on } \underline{C} \\ \text{the unique solution of (10)} & \text{on } C \\ \bar{\pi} & \text{on } \overline{C} \end{cases}$$

is the growth optimal fraction to be invested in S^1. *If* $\underline{C}$, *respectively* $\overline{C}$, *has positive measure then* $\widetilde{S}^1$, *respectively,* $\widetilde{S}^0$, *is a supermartingale which is not a local martingale. If C has measure T both* $\widetilde{S}^0$ *and* $\widetilde{S}^1$ *are local martingales.*

Proof. First note that Lemma 8.1 yields that π^* is an admissible strategy. Let Θ be an arbitrary portfolio in $\underline{\Theta}(S)$ with a corresponding fraction π invested in S^1. Itô's Lemma yields

$$\frac{d\left(\frac{1}{W_t^{\pi^*}}\right)}{\frac{1}{W_{t-}^{\pi^*}}} = -\pi_t^* \hat{\lambda}_t \int_E b_t^2(y)\phi(dy, dt) - \pi_t^* \int_E b_t(y)(\mu - \phi)(dy, dt)$$
$$+ \int_E \frac{b_t^2(y)\pi_t^{*2}}{1 + b_t(y)\pi_t^*}\mu(dy, dt),$$

which combined with the product formula yields the following drift term for $\frac{W_t^\pi}{W_t^{\hat{\pi}}}$:

$$\frac{W_{t-}^\pi}{W_{t-}^{\pi^*}}(\pi_t - \pi_t^*)\left(\hat{\lambda}_t \int_E b_t^2(y)\varphi_t(y)\kappa(dy) - \Pi_t(\pi_t^*)\right)dt. \tag{11}$$

From the last statement of Theorem 4.1 π^* is the GOP strategy if this quantity is negative for all π. This is true since $\Pi_t(\cdot)$ is increasing.

Note that it may well be that *both* of $\widetilde{S}^0$ and $\widetilde{S}^1$ are supermartingales—this could be the case for $\underline{\pi}_t < 0$ or $\bar{\pi}_t > 1$. □

A special case arises if we consider a defaultable security.

Corollary 8.3. *If* $\kappa(y \in E \mid b_t(y) = -1) > 0$ *and* $\hat{\lambda} \geq 0$ *then the set C has measure T and the solution* π^* *of* (10) *is the growth optimal fraction to be invested in* S^1.

Proof. In this case Π_t has a vertical asymptote for $\pi = 1$, $\lim_{\pi \to 1} \Pi_t(\pi) = \infty$, and since $\Pi_t(0) = 0$ there is a unique solution of Equation (10) since the left-hand side of Equation (10) is nonnegative. □

The above corollary implies that *if* a defaultable security has a slim chance of default, then the bankruptcy aversion of the growth-optimal investor ensures an optimum such that the inverse GOP is indeed a sigma martingale density. In contrast, if one considers a model where the loss given default has a thin tail of high losses and a positive market price of risk the *log*-investor goes as far as the admissibility constraints allow. In this case, we obtain a boundary solution and consequently the inverse of the GOP is only a supermartingale density. One could say that this constitutes a case where the market price of default risk is high compared to the potential losses.

9. CONCLUSION

The main purpose of this article was to characterize when asset pricing can be performed without the usual assumptions set forth in the literature. In order to accomplish this, we applied a generalized definition of a GOP. Such a generalization retains the usual GOP properties and the existence of such a portfolio is shown to be equivalent to the existence of a strict sigma martingale density. The proof relies on the duality result of [13] and an approximation procedure.

In the second part of the paper the market price of risk is introduced. This process is closely related to martingale densities and this relationship is used to derive a characterization of the growth optimal fraction under a set of sufficient conditions. This set of conditions can be translated into the dual language of [13], namely *if* the *log*-investor's optimizer corresponds to a sigma martingale density then this set of conditions completely characterizes the GOP. In the last section, the GOP characterization is applied to investigate when the inverse GOP is not a sigma martingale density.

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